British Mathematical Olympiad

Round 1

1997 to 2000

United Kingdom Mathematics Trust

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Organised by the **United Kingdom Mathematics Trust**

Contents

Introduction

This booklet contains the questions and solutions for the British Mathematical Olympiad Round 1 papers for the academic sessions 1996-97, 1997-98, 1998-99 and 1999-2000. BMO1 is a serious test in itself but it also acts as a gateway to BMO2 and further events leading to the selection of the United Kingdom team to compete in the International Mathematical Olympiad – held each summer. Each year, a booklet is produced covering both BMO1 and BMO2 and it is sent to schools who took part. In that booklet, there is a standard comment about the solutions:

> 'These solutions are the result of many hours of work by a large number of people. They have been subject to many drafts and revisions. As such, they do not resemble the first jottings, failed ideas and discarded pages of rough work with which any solution is started. Before looking at the solutions pupils and teachers are encouraged to make a good effort to solve the problems by themselves. Without wrestling with the problem oneself, it is hard to develop a feeling for the question, to understand where the difficulties lie and to appreciate why one method of attack is successful while another may fail. Reading these solutions without first attempting the question is unlikely to be of much benefit.
>
> Many thanks are due to the contestants and to the members of the setting committee who contributed variety, refinement, inventiveness and precision to these written solutions.'

It is hoped that the material in this booklet is tackled in the same spirit.

Further Resources

There is a wide range of books which can help students prepare for BMO 1. The most appropriate is, without doubt,

The Mathematical Olympiad Handbook - An Introduction to Problem Solving based on the First 32 British Mathematical Olympiads 1965-1996

by Tony Gardiner, Oxford University Press, ISBN 0 19 850105 6

Further useful books are

Elementary Number Theory

by David M. Burton, Allyn and Bacon, ISBN 0 205 06978 9

Student Problems from the Mathematical Gazette

The Mathematical Association (www.m-a.org.uk) ISBN 0 906588 49 9

It is hoped that, in due course, a section of the UKMT website will offer further suggestions.

The rubric for all the papers in this booklet is:

British Mathematical Olympiad

Round 1

Time allowed	*Three and a half hours.*
Instructions	• *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.* *Do not hand in rough work.* • *One* ***complete*** *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.* • *Each question carries 10 marks.* • *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.* • *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* ***left*** *hand corner and your name, initials and school in the top* ***right*** *hand corner.* • *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1, 2, 3, 4, 5 in order.* • *Staple all the pages neatly together in the top* ***left*** *hand corner.*

Wednesday, 15 January 1997

1. N is a four-digit integer, not ending in zero, and $R(N)$ is the four-digit integer obtained by reversing the digits of N; for example, $R(3275) = 5723$.
   Determine all such integers N for which $R(N) = 4N + 3$.

2. For positive integers n, the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is defined by
   $$a_1 = 1, \qquad a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + a_3 + \ldots + a_{n-1}), \qquad n > 1.$$
   Determine the value of a_{1997}.

3. The Dwarfs in the Land-under-the-Mountain have just adopted a completely decimal currency system based on the *Pippin*, with gold coins to the value of 1 *Pippin*, 10 *Pippins*, 100 *Pippins* and 1000 *Pippins*.
   In how many ways is it possible for a Dwarf to pay, in exact coinage, a bill of 1997 *Pippins*?

4. Let $ABCD$ be a convex quadrilateral. The midpoints of AB, BC, CD and DA are P, Q, R and S, respectively. Given that the quadrilateral $PQRS$ has area 1, prove that the area of the quadrilateral $ABCD$ is 2.

5. Let x, y and z be positive real numbers.
   (i) If $x + y + z \geqslant 3$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leqslant 3$?
   (ii) If $x + y + z \leqslant 3$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geqslant 3$?

Wednesday, 14 January 1998

1. A 5×5 square is divided into 25 unit squares. One of the numbers 1, 2, 3, 4, 5 is inserted into each of the unit squares in such a way that each row, each column and each of the two diagonals contains each of the five numbers once and only once. The sum of the numbers in the four squares immediately below the diagonal from top left to bottom right is called the *score*.
   Show that it is impossible for the score to be 20.
   What is the highest possible score?

2. Let $a_1 = 19$, $a_2 = 98$. For $n \geqslant 1$, define a_{n+2} to be the remainder of $a_n + a_{n+1}$ when it is divided by 100. What is the remainder when
$$a_1^2 + a_2^2 + \ldots + a_{1998}^2$$
   is divided by 8?

3. ABP is an isosceles triangle with $AB = AP$ and $\angle PAB$ acute. PC is the line through P perpendicular to BP, and C is a point on this line on the same side of BP as A. (You may assume that C is not on the line AB.) D completes the parallelogram $ABCD$. PC meets DA at M.
   Prove that M is the midpoint of DA.

4. Show that there is a unique sequence of positive integers (a_n) satisfying the following conditions:
$$a_1 = 1, \qquad a_2 = 2, \qquad a_4 = 12$$
$$a_{n+1}a_{n-1} = a_n^2 \pm 1 \qquad \text{for} \qquad n = 2, 3, 4, \ldots .$$

5. In triangle ABC, D is the midpoint of AB and E is the point of trisection of BC nearer to C. Given that $\angle ADC = \angle BAE$ find $\angle BAC$.

Wednesday, 13 January 1999

1. I have four children. The age in years of each child is a positive integer between 2 and 16 inclusive and all four ages are distinct. A year ago the square of the age of the oldest child was equal to the sum of the squares of the ages of the other three. In one year's time the sum of the squares of the oldest and the youngest will be equal to the sum of the squares of the other two children.

   Decide whether this information is sufficient to determine their ages uniquely, and find all possibilities for their ages.

2. A circle has diameter AB and X is a fixed point of AB lying between A and B. A point P, distinct from A and B, lies on the circumference of the circle. Prove that, for all possible positions of P,
$$\frac{\tan \angle APX}{\tan \angle PAX}$$
remains constant.

3. Determine a positive constant c such that the equation
$$xy^2 - y^2 - x + y = c$$
has precisely three solutions (x, y) in positive integers.

4. Any positive integer m can be written uniquely in base 3 form as a string of 0's, 1's and 2's (not beginning with a zero). For example
$$98 = (1 \times 81) + (0 \times 27) + (1 \times 9) + (2 \times 3) + (2 \times 1) = (10122)_3.$$
Let $c(m)$ denote the sum of the cubes of the digits of the base 3 form of m; thus, for instance
$$c(98) = 1^3 + 0^3 + 1^3 + 2^3 + 2^3 = 18.$$
Let n be any fixed positive integer. Define the sequence (u_r) by
$$u_1 = n \quad \text{and} \quad u_r = c(u_{r-1}) \quad \text{for } r \geqslant 2.$$
Show that there is a positive integer r for which $u_r = 1$, 2 or 17.

5. Consider all functions f from the positive integers to the positive integers such that
   (i) for each positive integer m, there is a unique positive integer n such that $f(n) = m$;
   (ii) for each positive integer n, we have
$$f(n+1) \text{ is either } 4f(n) - 1 \text{ or } f(n) - 1.$$
Find the set of positive integers p such that $f(1999) = p$ for some function f with properties (i) and (ii).

Wednesday, 12 January 2000

1. Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects angle PMR.

2. Show that, for every positive integer n,
$$121^n - 25^n + 1900^n - (-4)^n$$
is divisible by 2000.

3. Triangle ABC has a right angle at A. Among all points P on the perimeter of the triangle, find the position of P such that
$$AP + BP + CP$$
is minimized.

4. For each positive integer k, define the sequence $\{a_n\}$ by
$$a_0 = 1 \qquad \text{and} \qquad a_n = kn + (-1)^n a_{n-1} \qquad \text{for each } n \geqslant 1.$$
Determine all values of k for which 2000 is a term of the sequence.

5. The seven dwarfs decide to form four teams to compete in the Millennium Quiz. Of course, the sizes of the teams will not all be equal. For instance, one team might consist of Doc alone, one of Dopey alone, one of Sleepy, Happy and Grumpy as a trio, and one of Bashful and Sneezy as a pair. In how many ways can the four teams be made up? (The order of the teams or of the dwarfs within the teams does not matter, but each dwarf must be in exactly one of the teams.)
Suppose Snow White agreed to take part as well. In how many ways could the four teams then be formed?

Solutions to the 1997 paper

1. N is a four-digit integer, not ending in zero, and $R(N)$ is the four-digit integer obtained by reversing the digits of N; for example, $R(3275) = 5723$.

   Determine all such integers N for which $R(N) = 4N + 3$.

Solution 1

Let $N = 1000a + 100b + 10c + d$ whence $R(N) = 1000d + 100c + 10b + a$ and we have $1000d + 100c + 10b + a = 4000a + 400b + 40c + 4d + 3$ whence $996d + 60c = 390b + 3999a + 3$ or $332d + 20c = 130b + 1333a + 1$.

Now, (i) a is odd (else LHS even while RHS odd); and

(ii) if $a \geqslant 3$, we have LHS $\leqslant 3168$ (when $c = d = 9$) $< 4000 \leqslant$ RHS (when $b = 0$ and $a = 3$).

So it follows that $a = 1$ and $332d + 20c = 130b + 1334$ $\Rightarrow 166d + 10c = 65b + 667$ from which it follows that

(iii) b is also odd.

Then RHS $\equiv 2 \pmod{10} \Rightarrow$ LHS $\equiv 6d \equiv 2 \pmod{10}$ also $\Rightarrow d \equiv 2$ or $7 \pmod{10}$, and, since $1 \leqslant d \leqslant 9$, $d = 2$ or 7.

Case I: $d = 2 \Rightarrow 10c = 65b + 335 \Rightarrow 2c = 13b + 67$.
But $c \leqslant 9 \Rightarrow$ LHS $\leqslant 18 < 67 \leqslant$ RHS, so that there are no solutions in this case.

Case II: $d = 7 \Rightarrow 495 + 10c = 65b \Rightarrow 99 + 2c = 13b$.
Now $0 \leqslant c \leqslant 9 \Rightarrow$ LHS $= 99, 101, 103, \ldots, 117$ and, of these possibilities, only 117 (when $c = 9$) is a multiple of 13.

Thus, from Case II, we must have $a = 1$, $d = 7$, $c = 9$ and $b = 9$ (since $117 = 9 \times 13$) giving $N = 1997$ as the only possible solution, and it is easily checked that this does indeed satisfy the requirements of the problem.

Solution 2

Since $R(N) = 4N + 3$ is a four-digit number, $4N + 3 < 10000$, so that the first digit of N must either be 1 or 2. However, the first digit of N is the final digit of $R(N)$, which is odd. Thus the first digit of N is 1.

It follows that $4N + 3$ has final digit 1 and this requires N to have final digit 2 or 7. But $R(N) = 4N + 3 > 4000$ and so $R(N)$ cannot have first digit 2. Hence the final digit of N is 7.

We now have $7000 \leqslant 4N + 3 < 8000$, so that the first two digits of N are 17, 18 or 19. Of these, 18 can be ruled out immediately since it gives $R(N) = 7{*}81$ which leaves a remainder of 1 (instead of 3) upon division by 4. Thus $N = 17{*}7$ or $19{*}7$, corresponding to $4N = R(N) - 3 = 7{*}68$ or $7{*}88$. Recalling that N has final digit 7 (not 2), we see that its last two digits must be 17 or 67 in the first case, and 47 or 97 in the second.

For the resulting four possibilities: $N = 1717, 1767, 1947, 1997$ we have

$$4N + 3 = 6871,\ 7071,\ 7791,\ 7991$$

with, again, only the one possibility fitting the bill, namely $N = 1997$.

Note: In the third paragraph of this solution, another observation which could be used to fill in the $*$s (once we have arrived at $N = 17{*}7$ or $19{*}7$), is that $N + R(N)$ is divisible by 11. This is easily established:

$$(1000a + 100b + 10c + d) + (a + 10b + 100c + 1000d) = 1001(a + d) + 110(b + c)$$

and 1001 and 110 are each multiples of 11. [In fact, the same result holds whenever N has an even number of digits.] Then we require $5N + 3 \equiv 0 \pmod{11}$, i.e. $5N \equiv 8 \pmod{11}$. Multiplying by 9 throughout leads to $45N \equiv 72 \pmod{11}$, i.e. $N \equiv 6 \pmod{11}$. This reduces the possibilities to 1777 and 1997, which can be checked as before.

Remark: Toby Gee noted that, for each $k = 2, 3, \dots$, the k-digit number $N = 2 \times 10^{k-1} - 3$ is such that $R_k(N) = 4N + 3$, where $R_k(N)$ denotes the k-digit reverse of N.

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2. For positive integers n, the sequence $a_1, a_2, a_3, \dots, a_n, \dots$ is defined by

$$a_1 = 1, \qquad a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + a_3 + \ \dots\ + a_{n-1}), \qquad n > 1.$$

Determine the value of a_{1997}.

Solution 1

It is natural to start by working out a few initial values. We have

$$a_1 = 1 \text{ (given)},\ a_2 = 3,\ a_3 = 8,\ a_4 = 20,\ a_5 = 48,\ a_6 = 112,\ \dots .$$

Since we are asked for a_{1997}, which is going to be a very large number, a factorised form is likely to be the most practicable way of giving it. Factorising the first few terms leads to

$$a_1 = 2 \times \tfrac{1}{2}, a_2 = 3 \times 1, a_3 = 4 \times 2, a_4 = 5 \times 4, a_5 = 6 \times 8, a_6 = 7 \times 16, \dots$$

and it is possible to conjecture that $a_n = (n + 1)2^{n-2}$ for all positive integers n. This conjecture now needs to be proved.

The result is true for $n = 1$, from above. Assume the result is true for all $n = 1, 2, 3, \dots, k$. Then

$$a_{k+1} = \frac{k+2}{k}(a_1 + a_2 + a_3 + \ldots + a_k) \text{ from the given definition}$$
$$= \frac{k+2}{k}\{1 + 3 + 8 + \ldots + 2^{k-2}(k+1)\} = \frac{k+2}{k} S \text{ (say)}.$$

Now $\quad S = 1 + 3 + 8 + 20 + 48 + 112 + \ldots + 2^{k-2}(k+1)$

$\Rightarrow \quad 2S = \quad 2 + 6 + 16 + 40 + 96 + \ldots + 2^{k-2}k \quad + 2^{k-1}(k+1).$

Subtracting gives

$$\begin{aligned} S &= -1 - \{1 + 2 + 4 + 8 + 16 + \ldots + 2^{k-2}\} + k.2^{k-1} + 2^{k-1} \\ &= -1 - \{2^{k-1} - 1\} + k.2^{k-1} + 2^{k-1} \\ &= k.2^{k-1}. \end{aligned}$$

Then $a_{k+1} = \dfrac{k+2}{k}\left(k.2^{k-1}\right) = (k+2)2^{k-1}$, as required.

It now follows by strong induction that $a_n = (n+1)2^{n-2}$ for all positive integers n.

Then, finally, $a_{1997} = 1998 \times 2^{1995}$, or equivalent.

Solution 2

As an alternative to the initial approach in Solution 1, another natural thing to do is to experiment with the sequence-definition. For instance, we could note that

$$\begin{aligned} (n-1)a_n &= (n+1)\{(a_1 + a_2 + \ldots + a_{n-2}) + a_{n-1}\} \text{ for } n > 2 \\ &= (n+1)\left\{\frac{n-2}{n} a_{n-1} + a_{n-1}\right\} = (n+1)\left\{\frac{2n-2}{n} a_{n-1}\right\} \\ &= \frac{2(n-1)(n+1)}{n} a_{n-1} \end{aligned}$$

so that $a_n = \dfrac{2(n+1)}{n} a_{n-1}$, being true for $n > 1$ (checking directly in the case $n = 2$ confirms this).

We can now proceed in a number of different ways. Firstly, we could try and use this more concise relationship as part of an inductive proof. With the conjecture that

$$a_n = (n+1)2^{n-2} \text{ for all positive integers } n = 1, 2, 3, \ldots, k,$$

(and knowing it to be true for $n = 1, 2, 3, 4, 5, 6, \ldots$ as already determined), we can proceed to deduce that

$$a_{k+1} = \frac{2(k+2)}{k+1} a_k = \frac{2(k+2)}{k+1} \times (k+1)2^{k-2} = (k+2)2^{k-1}$$

and the result follows by induction, as before.

Alternatively, we can attempt to use the *first-order* recurrence definition to obtain the result directly. Now

$$a_n = \frac{2(n+1)}{n} a_{n-1} \text{ and } a_{n-1} = \frac{2(n-1+1)}{n-1} a_{n-2} \text{ together give}$$

$$\begin{aligned}
a_n &= \frac{2(n+1)}{n} \times \frac{2n}{n-1} a_{n-2} \\
&= \frac{2(n+1)}{n} \times \frac{2n}{n-1} \times \frac{2(n-1)}{n-2} a_{n-3} \\
&= \frac{2(n+1)}{n} \times \frac{2n}{n-1} \times \frac{2(n-1)}{n-2} \times \frac{2(n-2)}{n-3} a_{n-4} \\
&= \ldots \\
&= \frac{2(n+1)}{n} \times \frac{2n}{n-1} \times \frac{2(n-1)}{n-2} \times \ldots \times \frac{2(5)}{4} \times \frac{2(4)}{3} \times \frac{2(3)}{2} a_1 \\
&= 2^{n+1-2} \times \frac{(n+1)}{2} \times 1 \text{ (since } a_1 = 1) \\
&= 2^{n-2}(n+1) \text{, as before.}
\end{aligned}$$

Solution 3

Having derived, as above, the relation $a_n = \dfrac{2(n+1)}{n} a_{n-1}$, $(n > 1)$, we can proceed as follows. Rewrite this as $\dfrac{a_n}{n+1} = \dfrac{2a_{n-1}}{n}$ and define the sequence $\{b_n\}$ by $b_n = \dfrac{a_n}{n+1}$ for $n \geqslant 1$. Thus $b_1 = \frac{1}{2}$ and $b_n = 2b_{n-1}$ for $n > 1$.

We now have $b_n = 2b_{n-1} = 2^2 b_{n-2} = 2^3 b_{n-3} = \ldots = 2^{n-1} b_1 = 2^{n-2}$. Substituting back for a_n then gives $a_n = (n+1)2^{n-2}$, from which $a_{1997} = 1998 \times 2^{1995}$, as before.

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3. The Dwarfs in the Land-under-the-Mountain have just adopted a completely decimal currency system based on the *Pippin*, with gold coins to the value of 1 *Pippin*, 10 *Pippins*, 100 *Pippins* and 1000 *Pippins*.

   In how many ways is it possible for a Dwarf to pay, in exact coinage, a bill of 1997 *Pippins*?

Solution 1

Firstly. note that the 'extra' 7 Pippins can only be made up using 1P coins, so that the number of ways of making up 1997 Pippins is precisely the same as the number of ways of making up 1990 Pippins.

Using only 1P and 10P coins, a bill of 1990 Pippins can be paid using 0 or 1 or 2 or 3 or … or 199 10P coins: that is, in 200 ways.

If one 100P coin were used, then the remaining 1890 Pippins could be paid in 190 ways using 1P and 10P coins (by a similar argument).

If two 100P coins were used, the remaining 1790 Pippins could be paid in 180 ways, and so on. We see, then, that the number of ways of paying a bill of 1990 Pippins is

$$200 + 190 + 180 + \ldots + 10 = \frac{20}{2}(200 + 10) = 2100,$$

by the formula for the sum of the first n terms of an arithmetic progression.

Finally, if one 1000P coin were to be used, it can be shown similarly that the remaining 990 Pippins could be paid in

$$100 + 90 + \ldots + 10 = \frac{10}{2}(100 + 10) = 550 \text{ ways}.$$

Thus the total number of ways of paying (in exact coinage) a bill of 1997 Pippins is 2650.

Solution 2

The following approach is essentially the same as the first one, but more obviously 'builds up' towards the solution by examining the simpler cases where, firstly, only 1P coins are available; then only 1P and 10P coins are available: then 1P, 10P and 100P coins are available; finally introducing the 1000P coin in addition to those of lower value. Also, there is an attempt to approach the matter in a way which can be extended to a more general situation.

1997P can be paid in one way using only 1P coins.

1997P can be paid in 200 ways using only 1P and 10P coins. since we can use 0 or 1 or 2 or … or 199 10P coins, making up the remaining amount in each case with 1P coins.

In fact, it is easily seen that if there is a bill of N Pippins. then

$T(N)$ = the number of ways of paying N Pippins using 1P and 10P coins only

$$= \left[\frac{N}{10}\right] + 1.$$

where $[x]$ denotes the greatest integer less than or equal to x; this is so since we can use

$$0 \text{ or } 1 \text{ or } \ldots \text{ or } \left[\frac{N}{10}\right] \text{ 10P coins}$$

with the remaining 'units' amounts necessarily being paid by 1P coins.

Alternatively, if $N = 10x + y\ (x \geqslant 0,\ 0 \leqslant y \leqslant 9)$, then $T(N) = x + 1$.

If we now include 100P coins, we can choose to pay with

0×100P coins in $T(1997)$ ways
1×100P coins in $T(1897)$ ways
2×100P coins in $T(1797)$ ways
... ...
19×100P coins in $T(97)$ ways

giving, altogether, $200 + 190 + 180 + \dots + 20 + 10 = 2100$ ways.

If we let $H(N)$ = the number of ways of paying N Pippins using 1P, 10P and 100P coins only, then we see that

$$H(N) = T(N) + T(N - 100) + T(N - 200) + \dots + T(N - 100m)$$

where m is determined by the greatest multiple of 100 which can be subtracted from N (without leaving a negative quantity); that is, $m = \left[\frac{N}{100}\right]$. Thus,

$$H(N) = \sum_{i=0}^{[N/100]} T(N - 100i).$$

Finally, allowing for the inclusion of 1000P coins, we can choose

0×1000P coins in 2100 ways (as above)

or 1×1000P coin.

For this latter case we could then choose

0×100P coins in $T(997)$ ways
1×100P coins in $T(897)$ ways
...
9×100P coins in $T(97)$ ways

giving a further $100 + 90 + \dots + 10 = 550$ ways.

The required answer is thus 2650 ways.

Solution 3

Elaborating on the above approaches, let us define the function $f(n, k)$, for integers $n, k \geqslant 0$, to be the number of ways of paying a bill of n Pippins (in exact coinage) using only 1P, 10P, 100P, ..., 10^kP coins. With this notation, we wish to evaluate $f(1997, 3)$.

In general, $f(n, k) = \sum_{i=0}^{[n/10^k]} f\big((n - 10^k i), k - 1\big)$, as we have already seen in the case $k = 2$.

Now $f(n, 0) = 1$ since there is no choice:

$$f(n, 1) = \sum_{i=0}^{[n/10]} 1 = \left[\frac{n}{10}\right] + 1;$$

$$f(n,\ 2) = \sum_{i=0}^{[n/100]} \left\{ \left[\frac{n - 100i}{10} \right] + 1 \right\} = \sum_{i=0}^{[n/100]} \left\{ \left[\frac{n}{10} \right] - 10i + 1 \right\}$$

$$= \sum_{i=0}^{[n/100]} \left\{ \left[\frac{n}{10} \right] + 1 \right\} - 10 \sum_{i=0}^{[n/100]} i$$

$$= \left\{ \left[\frac{n}{10} \right] + 1 \right\} \times \left\{ \left[\frac{n}{100} \right] + 1 \right\} - 10 \times \frac{1}{2} \left\{ \left[\frac{n}{100} \right] \right\} \left\{ \left[\frac{n}{100} \right] + 1 \right\}$$

$$= \left\{ \left[\frac{n}{100} \right] + 1 \right\} \times \left\{ \left[\frac{n}{10} \right] - 5 \left[\frac{n}{100} \right] + 1 \right\}.$$

Thus $f(1997,\ 2) = (19 + 1)(199 - 5 \times 19 + 1) = 20 \times 105 = 2100$
and $f(997,\ 2) = (9 + 1)(99 - 5 \times 9 + 1) = 10 \times 55 = 550.$
so that $f(1997,\ 3) = f(1997,\ 2) + f(997,\ 2) = 2650.$

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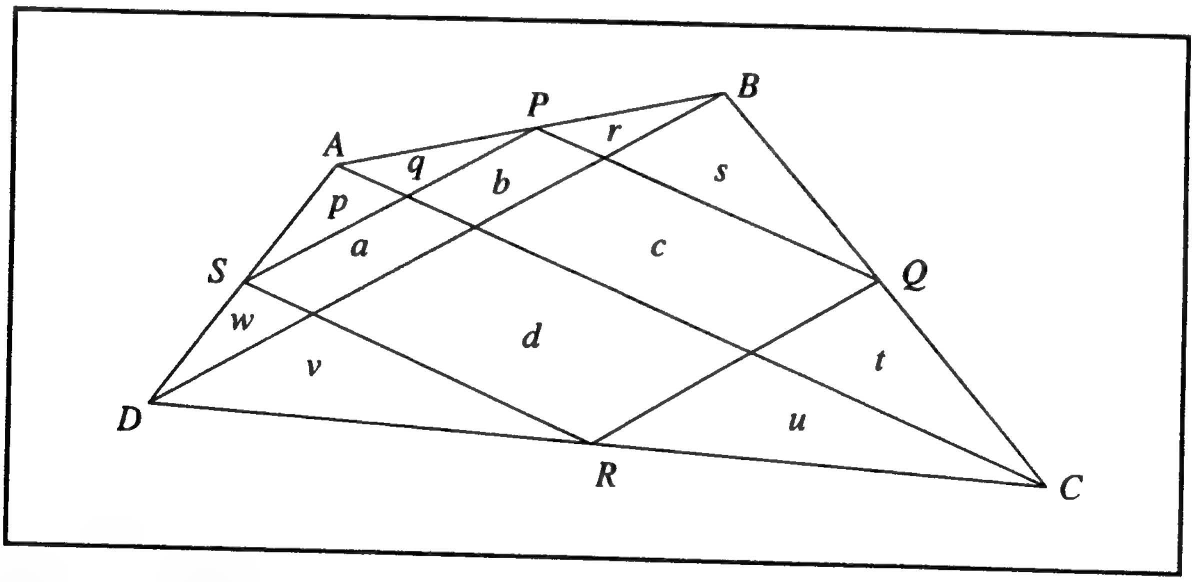
4. Let $ABCD$ be a convex quadrilateral. The midpoints of AB, BC, CD and DA are P, Q, R and S, respectively. Given that the quadrilateral $PQRS$ has area 1, prove that the area of the quadrilateral $ABCD$ is 2.

Solution 1



Draw in the diagonals AC and BD.

Consider triangles APS and ABD. Since P is the midpoint of AB and S is the midpoint of AD, it follows that PS is parallel to, and half the length of, BD. Thus triangles $\left.\begin{matrix} APS \\ ABD \end{matrix}\right\}$ are similar with corresponding lengths in the ratio $1 : 2$.

It follows (for example, from the usual '$\Delta = \frac{1}{2}$ ×base × height' formula), that

$$\text{area } \Delta APS = \tfrac{1}{4}(\text{area } \Delta ABD).$$

Similarly, area $\Delta BPQ = \frac{1}{4}$(area ΔBAC), area $\Delta CQR = \frac{1}{4}$(area ΔCBD) and

area $\Delta DRS = \frac{1}{4}(\text{area } \Delta DCA)$.

Using the areas as indicated on the above diagram, we obtain the following:

$$p + q = \tfrac{1}{4}(p + q + w + a + b + r)$$
$$r + s = \tfrac{1}{4}(r + s + q + b + c + t)$$
$$t + u = \tfrac{1}{4}(t + u + s + c + d + v)$$
$$\text{and } v + w = \tfrac{1}{4}(v + w + u + d + a + p).$$

Adding these four equations gives

$$(p + q + r + s + t + u + v + w)$$
$$= \tfrac{1}{4}\{2(p + q + r + s + t + u + v + w) + 2(a + b + c + d)\}$$

whence $p + q + r + s + t + u + v + w = a + b + c + d$ so that area $ABCD = 2 \times$ area $PQRS$. Given that area $PQRS = 1$, we immediately obtain area $ABCD = 2$.

Note: There is a standard result that, for any quadrilateral $ABCD$, not even necessarily convex, the quadrilateral formed by joining the midpoints of the sides is always a parallelogram. This result can be established using the above approach: the crucial starting point again being to draw in the diagonals of $ABCD$. There is also a very nice, simple proof using vectors.

Solution 2

A more concise approach might run as follows.

Again, noting the similarity of the various pairs of triangles, we have

area $\Delta APS = \frac{1}{4}(\text{area } \Delta ABD)$ and, similarly, area $\Delta CRQ = \frac{1}{4}(\text{area } \Delta CDB)$.

Adding these together gives

area ΔAPS + area $\Delta CRQ = \frac{1}{4}(\text{area } \Delta ABD + \text{area } \Delta CDB) = \frac{1}{4}(\text{area } ABCD)$.

Similarly, area ΔBPQ + area $\Delta DRS = \frac{1}{4}(\text{area } ABCD)$.

Thus the area of the four outer triangles is equal to $\frac{1}{2}$ the area of $ABCD$, from which it follows that the area of quadrilateral $PQRS$ is also equal to $\frac{1}{2}$ the area of $ABCD$. Given that $PQRS$ has area 1, then $ABCD$ has area 2.

Solution 3

The following approach avoids the purely geometrical set-up of Solution 1.

Using the well-known formula for the area of a triangle, namely $\Delta = \frac{1}{2}ab \sin C$, we have

area $\Delta APS = \frac{1}{2}AP.AS \sin A = \frac{1}{2}(\frac{1}{2}AB).(\frac{1}{2}AD) \sin A = \frac{1}{4}(\text{area } \Delta ABD)$, etc.

and the solution now follows exactly the pattern of Solution 2 above.

5. Let x, y and z be positive real numbers.

(i) If $x + y + z \geqslant 3$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leqslant 3$?

(ii) If $x + y + z \leqslant 3$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geqslant 3$?

(i) Experimentation should lead quickly to the answer NO. In order to disprove the result we simply have to produce a counterexample: that is, a set of values of x, y, z satisfying the given conditions but for which the stated result does not hold. There are many possible ways to do this; three such ways are given here.

Take, for instance, $x = 3, y = z = \frac{1}{n}$ for n a positive integer $\geqslant 2$.

Then $x + y + z = 3 + \frac{2}{n} > 3$ while $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{3} + 2n > 3$ also, since $n \geqslant 2$.

Alternatively, try $x = 1, y = 2 - \varepsilon, z = \varepsilon$ where $\varepsilon > 0$ is very small (and < 2 in particular).

Then $x + y + z = 3$ but $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \rightarrow \infty$ as $\varepsilon \rightarrow 0$ since $\frac{1}{z} \rightarrow \infty$.

Or you could choose $x = 1, y = 2, z = \frac{1}{2}$.

Then $x + y + z = 3\frac{1}{2} > 3$ while $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3\frac{1}{2} > 3$ also.

(ii) Experimentation suggests that the answer is YES. Now we must prove that this is so. We shall first establish the following result:

Lemma For positive real numbers x, y, z

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geqslant 9.$$

with equality if and only if $x = y = z$.

Proof 1: By the Arithmetic Mean – Harmonic Mean Inequality for positive reals x, y, z

$$\frac{x + y + z}{3} \geqslant \frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}},$$

with equality if and only if $x = y = z$. The Lemma follows immediately.

Proof 2: Applying Cauchy's Inequality (the Cauchy–Schwarz Inequality) for positive x, y, z, to the sets $\{\sqrt{x}, \sqrt{y}, \sqrt{z}\}$ and $\left\{\frac{1}{\sqrt{x}}, \frac{1}{\sqrt{y}}, \frac{1}{\sqrt{z}}\right\}$.

$$\left(\sqrt{x}.\frac{1}{\sqrt{x}} + \sqrt{y}.\frac{1}{\sqrt{y}} + \sqrt{z}.\frac{1}{\sqrt{z}}\right)^2 \leqslant (x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right),$$

with equality if and only if $\sqrt{x} : \frac{1}{\sqrt{x}} = \sqrt{y} : \frac{1}{\sqrt{y}} = \sqrt{z} : \frac{1}{\sqrt{z}}$, i.e. $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$.

The Lemma again follows.

Proof 3: For positive reals x, y, z,

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 3 + \left(\frac{x}{y} + \frac{y}{x}\right) + \left(\frac{y}{z} + \frac{z}{y}\right) + \left(\frac{z}{x} + \frac{x}{z}\right) \geqslant 3 + 2 + 2 + 2 = 9.$$

by the Arithmetic Mean – Geometric Mean Inequality, which establishes the Lemma.

We can now proceed:

$$3 \geqslant x + y + z \Rightarrow 3\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geqslant (x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geqslant 9,$$

from which it follows that $\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geqslant 3.$

Solutions to the 1998 paper

1. A 5×5 square is divided into 25 unit squares. One of the numbers 1, 2, 3, 4, 5 is inserted into each of the unit squares in such a way that each row, each column and each of the two diagonals contains each of the five numbers once and only once. The sum of the numbers in the four squares immediately below the diagonal from top left to bottom right is called the *score*.

   Show that it is impossible for the score to be 20.

   What is the highest possible score?

Solution

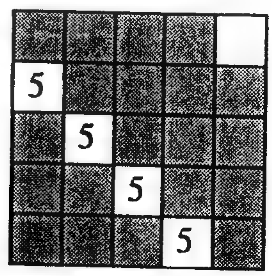
We start by naming the diagonal from top left to bottom right the *major diagonal*, the four squares below the major diagonal the *scoreline*, and the diagonal from bottom left to top right the *minor diagonal.* We will call an arrangement of the numbers 1,2,3,4,5 in the square *valid* if it satisfies all the criteria of the question.

We also make the following observations:

- There is a great deal of symmetry amongst valid arrangements. If any valid arrangement is reflected about the middle column, the middle row, the major diagonal or the minor diagonal then the resulting arrangement will also be valid. If a valid arrangement is rotated about the centre square through 90° then the resulting arrangement is valid. If the numbers are permuted amongst themselves, for example if the 1s are switched with the 5s, then we shall have another valid arrangement.
- The score itself has reflective symmetry about the minor diagonal. Any valid arrangement of numbers in the square reflected in the minor diagonal will give a valid arrangement with the same score.

To show that a score of 20 is impossible

To achieve a score of 20 all four entries on the scoreline must be 5s. The diagram shows four 5s placed in the scoreline. We cannot place a 5 on the shaded squares since this would violate the condition that there is only one 5 in any row or column. The condition that there should be exactly one 5 on the major diagonal cannot be met. Hence a score of 20 is not possible.



To find the highest possible score

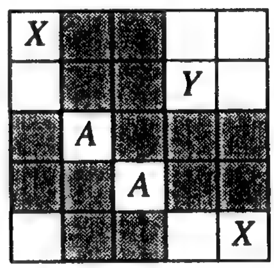
Solution 1

The highest possible score is 17. To prove this we eliminate the possibility of scores 20, 19 and 18 and then find an example of a square with score 17. To eliminate the possibilities of scores of 19 and 18 systematically we first consider how these totals can be achieved as the sum of four numbers from 1, 2, 3, 4, 5:

$$19 = 5 + 5 + 5 + 4;$$
$$18 = 5 + 5 + 5 + 3 = 5 + 5 + 4 + 4.$$

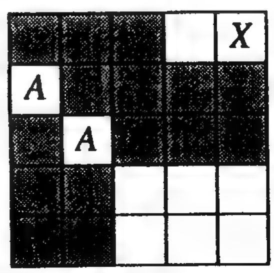
The numbers making up these scores could appear in any order on the scoreline. To avoid repetition of argument we need only consider the possibilities for placing a given number, A say, twice on the scoreline. A number appearing three times on the scoreline is a special case of this.

(i) *It is not possible to have the same number repeated in the middle two squares of the scoreline.*



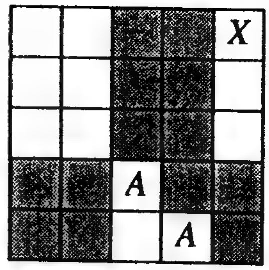
The diagram shows the placement of the 2 As in the scoreline. An A cannot be placed in any of the shaded squares without the number of As in a row or column exceeding 1. An A must be placed in the major diagonal. There are only two possibilities, the top left and bottom right squares labelled X in the diagram. These configurations are equivalent by symmetry about the minor diagonal. Without loss of generality assume that an A is placed in the top left corner. Now no A can be placed in any other corner in accordance with the criteria. Hence the A of the minor diagonal must be placed in the position marked with Y. Once this A is placed then there is no square left to put the remaining A without violating the conditions of the question.

(ii) *If the same number appears in the first two entries of the scoreline then that same number must appear in the top right-hand square.*



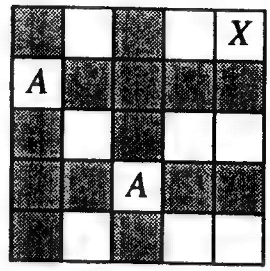
The diagram shows the placement of the 2 As in the scoreline. An A cannot be placed in any of the shaded squares without the number of As in a row or column exceeding 1. An A must be placed in the minor diagonal. There is only one place for it to go, indicated with an X.

By symmetry about the minor diagonal we have

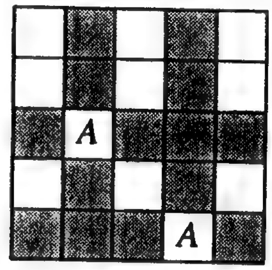


(iii) *If the same number appears in the last two entries of the scoreline then that same number must appear in the top right-hand square.*

(iv) *The same number cannot appear twice in the scoreline separated by only one square.*



We start by considering the case shown in the diagram to the right. There must be an *A* on the minor diagonal. It must go in the top right, indicated with an *X*. However placing an *A* here excludes the possibility of placing an *A* on the major diagonal.



By symmetry about the minor diagonal the case shown on the right is also excluded.

A Score of 19 is not possible

To score 19 there must be three 5s and one 4 on the scoreline. These numbers can be arranged in the orders 5554, 5455, 5545, 5554. By (iv) none of these arrangements is possible.

A Score of 18 is not possible

A valid arrangement with three 5s and one 3 on the scoreline is not possible for the same reasons as three 5s and one 4.

There are 6 possible arrangements of two 5s and two 4s on the scoreline. These can be eliminated as follows.

4455, 5544 – By (ii) and (iii) both a 4 and a 5 must appear in the top right square. But any square has only one entry so that this is impossible.

5445, 4554 – By (i) these arrangements are not possible.

5454, 4545 – By (iv) both of these arrangements are impossible.

A Score of 17 is possible

The square on the right shows a valid arrangement with a score of 17.

1	5	4	3	2
5	3	2	4	1
2	4	5	1	3
4	1	3	2	5
3	2	1	5	4

Solution 2 (W. Webb, King Edward's School, Birmingham & Jenny Wilson, Sutton High School)

The method of solution is to find the possible structure of any valid arrangement up to reflections and permutation of the numbers.

B				C
		A		
E				D

Let the number in the middle be denoted by the letter A. The numbers in the four comers must all be distinct from A since otherwise the number of As in the major and minor diagonals would exceed one. Furthermore, they must be distinct from each other since they share a row, column, major diagonal or minor diagonal with each other. Without loss of generality denote the entries in the corners clockwise from top left by B, C, D, E. At this stage we know that the arrangement looks like the square shown above on the right.

Now consider the major diagonal. Two numbers, C and E, occupy the two remaining squares. There is reflective symmetry about the minor diagonal so we may assume that the major diagonal entries appear in the order B, C, A, E, D. (If the order is B, E, A, C, D then reflect the arrangement in the minor diagonal and relabel B and D.) Now there are two possibilities for placing the remaining B and D in the minor diagonal. These are shown below and labelled (p) and (q) for reference.

(p)

B				C
	C		D	
		A		
	B		E	
E				D

(q)

B				C
	C		B	
		A		
	D		E	
E				D

Now we fill in the values of the remaining squares of (p) and (q) as they are determined by the criteria of the question. We start with square (p).

B			X	C
X	C		D	
		A		
	B		E	X
E	X			D

The value of each X in the square on the right must be A since a B, C, D and E all appear in the same row or column.

Once these As have been put in place there are two possibilities, D or E, for the top row second column entry. Once we are committed to one of these then the other entries are determined.

B	D	E	A	C
A	C	B	D	
		A		
	B		E	A
E	A			D

For example if the top row second column entry is D, then the top row third column entry is E, and the second row middle column entry is B, etc ....

Proceeding in this way, we arrive at two possibilities for valid arrangements from square (p).

B	*D*	*E*	*A*	*C*
A	*C*	*B*	*D*	*E*
D	*E*	*A*	*C*	*B*
C	*B*	*D*	*E*	*A*
E	*A*	*C*	*B*	*D*

B	*E*	*D*	*A*	*C*
A	*C*	*E*	*D*	*B*
C	*D*	*A*	*B*	*E*
D	*B*	*C*	*E*	*A*
E	*A*	*B*	*C*	*D*

With square (q) there are two possibilities, A or E, for the top row second column entry. Once we are committed to one of these choices then the rest of the values of the squares are determined.

For example if we suppose that it is an A than the bottom row second column entry must be a B, etc.

B				*C*
	C		*B*	
		A		
	D		*E*	
E				*D*

In this way we arrive at two possibilities for valid arrangements from square (q)

B	*A*	*E*	*D*	*C*
A	*C*	*D*	*B*	*E*
D	*E*	*A*	*C*	*B*
C	*D*	*B*	*E*	*A*
E	*B*	*C*	*A*	*D*

B	*E*	*D*	*A*	*C*
D	*C*	*E*	*B*	*A*
C	*B*	*A*	*D*	*E*
A	*D*	*C*	*E*	*B*
E	*A*	*B*	*C*	*D*

We have shown that there are essentially (up to permutation of numbers and reflections) four possible valid arrangements of numbers. These either have four distinct letters on the scoreline or one letter repeated on the scoreline with two distinct other letters. The highest score will be achieved by assigning the highest number, 5, to the repeated letter in the scoreline and the next highest numbers, 4 and 3, to the two remaining letters in the scoreline.

This gives a highest score of 17.

2. Let $a_1 = 19$, $a_2 = 98$. For $n \geqslant 1$, define a_{n+2} to be the remainder of $a_n + a_{n+1}$ when it is divided by 100. What is the remainder when

$$a_1^2 + a_2^2 + \ldots + a_{1998}^2$$

is divided by 8?

Solution

In this solution we use the established mathematical notation $m \equiv r \pmod n$ to mean that $m - r$ is exactly divisible by n, where n is a positive integer (thought of as greater than 1) and m, r are any integers. This relationship is equivalent to $m = r + kn$, for some integer k.

It is easy to check that if $a \equiv b \pmod n$ and $c \equiv d \pmod n$ then $a + c \equiv b + d \pmod n$. Indeed, if $a = b + pn$ and $c = d + qn$ then $a + c = b + d + (p + q)n$, where a, b, c, d, p and q are integers.

The question can now be restated as follows. If $a_1 = 19$, $a_2 = 98$ and $a_{n+2} \equiv a_n + a_{n+1} \pmod{100}$ find $a_1^2 + a_2^2 + \ldots + a_{1998}^2 \pmod 8$.

We first note that if $a \equiv b \pmod{100}$ then $a \equiv b \pmod 4$ since if $a = b + 100k$ then $a = b + 4 \times (25k)$. Hence $a_{n+2} \equiv a_n + a_{n-1} \pmod 4$.

Next we note that if $a \equiv b \pmod 4$ then $a^2 \equiv b^2 \pmod 8$, since if $a = b + 4k$ then $a^2 = b^2 + 8bk + 16k^2 = b^2 + 8 \times (bk + 2k^2)$.

Hence we need only consider the sequence $a_i \pmod 4$, $a_1 \equiv 3 \pmod 4$ and $a_2 \equiv 2 \pmod 4$. From the relationship $a_{n+2} \equiv a_n + a_{n+1} \pmod 4$ we get $a_3 \equiv 1 \pmod 4$, $a_4 \equiv 3 \pmod 4$, $a_5 \equiv 0 \pmod 4$, $a_6 \equiv 3 \pmod 4$, $a_7 \equiv 3 \pmod 4$ and $a_8 \equiv 2 \pmod 4$. We now have a pair of consecutive numbers (3, 2) repeated in the sequence. The next term in the sequence $a_i \pmod 4$ is found from the sum of the previous two numbers so the pattern 3, 2, 1, 3, 0, 3 must be repeated indefinitely.

We have established that, for any integer n, $a_{6n+1} \equiv 3 \pmod 4$, $a_{6n+2} \equiv 2 \pmod 4$, $a_{6n+3} \equiv 1 \pmod 4$, $a_{6n+4} \equiv 3 \pmod 4$, $a_{6n+5} \equiv 0 \pmod 4$, and $a_{6n+6} \equiv 3 \pmod 4$. Thus

$$\begin{aligned} a_{6n+1}^2 + a_{6n+2}^2 + a_{6n+3}^2 + a_{6n+4}^2 + a_{6n+5}^2 + a_{6n+6}^2 &\equiv 3^2 + 2^2 + 1^2 + 3^2 + 0^2 + 3^2 \pmod 8 \\ &\equiv 0 \pmod 8. \end{aligned}$$

Now 1998 is a multiple of 6 ($1998 = 333 \times 6$). Therefore

$$\begin{aligned} a_1^2 + a_2^2 + \ldots + a_{1998}^2 &\equiv \left(a_1^2 + a_2^2 + \ldots + a_6^2\right) + \left(a_7^2 + \ldots + a_{12}^2\right) + \ldots + \left(a_{1993}^2 + \ldots + a_{1998}^2\right) \\ &\equiv 0 + 0 + \ldots + 0 \pmod 8 \\ &\equiv 0 \pmod 8. \end{aligned}$$

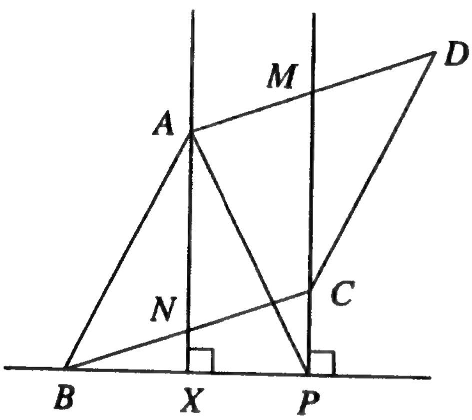
The remainder when $a_1^2 + a_2^2 + \ldots + a_{1998}^2$ is divided by 8 is 0.

3. ABP is an isosceles triangle with $AB = AP$ and $\angle PAB$ acute. PC is the line through P perpendicular to BP, and C is a point on this line on the same side of BP as A. (You may assume that C is not on the line AB.) D completes the parallelogram $ABCD$. PC meets DA at M.

Prove that M is the midpoint of DA.

We use $|AB|$ to denote the length of the line AB.

Solution 1



We are given that $|AB| = |AP|$; $|AD| = |BC|$ and $AD // BC$; $|AB| = |CD|$ and $AB // CD$.

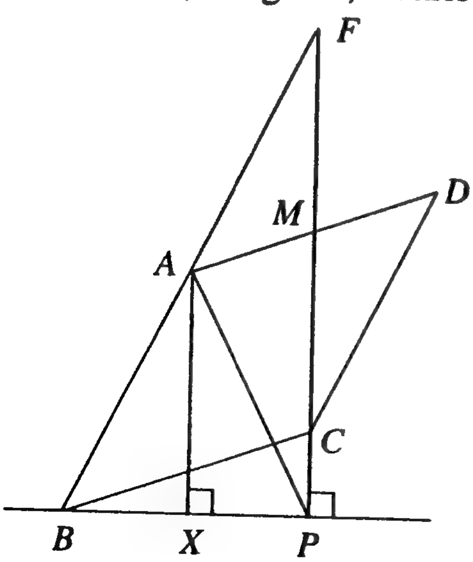
Let X be the midpoint of BP. So $\angle BXA = \angle PXA = 90°$ and $AX // PM$. Let N be the point of intersection of the lines AX and BC.

Triangles BXN and BPC are similar ($\angle BXN$ and $\angle BPC$ are both right angles, $\angle NBX$ and $\angle CBP$ are the same angle at B) and $|BX| = \frac{1}{2}|BP|$.

Since corresponding sides are in the same proportion it follows that $|BN| = \frac{1}{2}|BC|$. Hence $|NC| = |BC| - |BN| = \frac{1}{2}|BC|$.

Now $ANCM$ is a parallelogram since the opposite pairs of sides are parallel. Therefore $|AM| = |NC| = \frac{1}{2}|BC| = \frac{1}{2}|AD|$ and M is the midpoint of AD.

Solution 2 (Liang Mu, Henrietta Barnett School)



Extend the lines BA and PM and label their intersection F. Let X be the midpoint of BP.

Since XA and PF are parallel, triangles BXA and BPF are similar. Now X is the midpoint of BP so the ratio of the lengths of corresponding sides is $1:2$. Therefore $|AF| = |AB|$ and, since $ABCD$ is a parallelogram, $|AF| = |DC|$.

Now $\angle MFA = \angle MCD$ and $\angle MAF = \angle MDC$ (alternate angles) so triangles MAF and MCD are congruent (ASA). Hence the corresponding sides MA and MD have the same length and M is the midpoint of AD.

Solution 3 (J.R. Monroe, Westminster School & T. Barnet-Lamb, Culford School)

Introduce a coordinate system with B as the origin, BP as the x-axis and, without loss of generality, let P have coordinates (2, 0).

Point A is equidistant from B and P. Hence A lies on the line $x = 1$, the perpendicular bisector of BP. Let A have coordinates $(1, a)$.

Line PC is perpendicular to BP. Hence C lies on the line $x = 2$. Let C have coordinates $(2, c)$.

Now $ABCD$ is a parallelogram so that $\overrightarrow{BA} = \overrightarrow{CD}$ and the coordinates of D are $(3, a + c)$. The midpoint of AD is $(2, a + \frac{c}{2})$, which lies on the line $x = 2$. Hence the midpoint of AD is the point M, which lies at the intersection of the line AD and the extension of the line PC.

---

4. Show that there is a unique sequence of positive integers (a_n) satisfying the following conditions:

$$a_1 = 1, \qquad a_2 = 2, \qquad a_4 = 12$$

$$a_{n+1}a_{n-1} = a_n^2 \pm 1 \qquad \text{for} \qquad n = 2, 3, 4, \ldots .$$

Solution 1

We first consider the value of a_3.

From $a_3a_1 = a_2^2 \pm 1$, a_3 can be either 5 or 3.

From $a_4a_2 = a_3^2 \pm 1$, we get $24 = a_3^2 \pm 1$. It follows that $a_3 = 5$ is the only possible integer value.

Investigating the first few terms of the sequence we find that $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, $a_4 = 12$, $a_5 = \dfrac{12^2+1}{5} = 29$, $a_6 = \dfrac{29^2-1}{12} = 70$, $a_7 = \dfrac{70^2+1}{29} = 169$, $a_8 = \dfrac{169^2-1}{70} = 408, \ldots.$

It appears from these terms that

- the +, − signs are alternating so that $a_{2k+1} = \dfrac{a_{2k}^2+1}{a_{2k-1}}$ and $a_{2k+2} = \dfrac{a_{2k+1}^2-1}{a_{2k}}$;
- the terms of the sequence (a_n) are increasing in size;
- the sequence obeys the recurrence relation $a_{n+1} = 2a_n + a_{n-1}$, $(n \geqslant 2)$.

To prove the uniqueness of the sequence, we first show that it must be increasing.

Lemma 1: For $n \geqslant 2, a_{n+1} > a_n > 1$.

Proof

The proof is by induction.

We note that $a_3 > a_2 > 1$ so that the statement is true when $n = 2$. Let k be a value for which $a_{k+1} > a_k > 1$ so $a_{k+1} - a_k \geqslant 1$ and $a_{k+1} \geqslant 2$ since a_{k+1} and a_k are integers. We consider $a_{k+2} - a_{k+1}$.

$$\begin{aligned} a_{k+2} - a_{k+1} &= \frac{a_{k+1}^2 \pm 1}{a_k} - a_{k+1} \\ &= \frac{a_{k+1}(a_{k+1} - a_k) \pm 1}{a_k} > 0. \end{aligned}$$

Hence $a_{k+2} > a_{k+1} > 1$.

The statement is true when $n = 2$ and we have shown that if it is true when $n = k$ then it is true when $n = k + 1$. By induction the statement is true for all integers $n \geqslant 2$.

From Lemma 1 above, all terms in the sequence after the second are greater than 2. Hence, when $n \geqslant 4$, $a_n^2 + 1$ and $a_n^2 - 1$, which have a difference of 2, cannot both be divisible by a_{n-1}. Therefore the sequence, if it exists, is unique because we cannot take *both* of + and −.

It suffices now to show that the sequence defined by the recurrence relation $a_{n+1} = 2a_n + a_{n-1}$, $a_1 = 1$, $a_2 = 2$, which is a sequence of integers, coincides with the given sequence. Once again the proof is by induction.

Lemma 2: The sequence defined by the recurrence relation $a_{n+1} = 2a_n + a_{n-1}$, $a_1 = 1, a_2 = 2$ satisfies the relation $a_{n+1}a_{n-1} = a_n^2 \pm 1$, $(n \geqslant 2)$.

Proof

We have already seen that $a_3 = 5$ satisfies $a_3a_1 = a_2^2 + 1$ so the statement is true when $n = 2$.

Suppose that $a_{k+1}a_{k-1} = a_k^2 \pm 1$. Consider $a_{k+2}a_k$.

Since $a_{k+2} = 2a_{k+1} + a_k$ and $2a_k = a_{k+1} - a_{k-1}$.

$$\begin{aligned} a_{k+2}a_k &= (2a_{k+1} + a_k)a_k \\ &= 2a_{k+1}a_k + a_k^2 \\ &= a_{k+1}(a_{k+1} - a_{k-1}) + a_k^2 \\ &= a_{k+1}^2 - a_{k+1}a_{k-1} + a_k^2 \\ &= a_{k+1}^2 - (a_k^2 \pm 1) + a_k^2 \\ &= a_{k+1}^2 \mp 1. \end{aligned}$$

We have shown above that if the sequence satisfies $a_{n+1}a_{n-1} = a_n^2 \pm 1$ when $n = k$ then it satisfies $a_{n+1}a_{n-1} = a_n^2 \pm 1$ when $n = k + 1$. Since the sequence satisfies $a_{n+1}a_{n-1} = a_n^2 \pm 1$ when $n = 2$ it follows by induction that the sequence satisfies $a_{n+1}a_{n-1} = a_n^2 \pm 1$ for all integers $n \geqslant 2$.

Incidentally we have also shown that the negative and positive signs alternate.

Hence the sequence defined by the recurrence relation $a_{n+1} = 2a_n + a_{n-1}$, $a_1 = 1$, $a_2 = 2$ is the unique sequence of positive integers (a_n) satisfying the conditions $a_1 = 1, a_2 = 2, a_4 = 12$, and $a_{n+1}a_{n-1} = a_n^2 \pm 1$ for $n \geqslant 2$.

Solution 2

Establish that there can be at most one sequence of integers satisfying the conditions of the question as above. Now define a sequence (a_n) by $a_1 = 1$, $a_2 = 2$, $a_{2n+1} = \dfrac{a_{2n}^2 + 1}{a_{2n-1}}$ and $a_{2n+2} = \dfrac{a_{2n+1}^2 - 1}{a_{2n}}$, $(n \geqslant 1)$, and prove by induction that the recurrence relations $a_{2n+1} = 2a_{2n} + a_{2n-1}$ and $a_{2n+2} = 2a_{2n+1} + a_{2n}$, $(n \geqslant 1)$, are satisfied. The sequence is consequently a sequence of integers. The inductive step is as follows.

Assume that the sequence satisfies the recurrence relations for $n = k$. In particular we assume that $a_{2k+1} = 2a_{2k} + a_{2k-1}$ and $a_{2k+2} = 2a_{2k+1} + a_{2k}$. First consider a_{2k+3}.

$$a_{2k+3} = \frac{a_{2k+2}^2 + 1}{a_{2k+1}} = \frac{(2a_{2k+1} + a_{2k})^2 + 1}{a_{2k+1}} = 4a_{2k+1} + 4a_{2k} + \frac{a_{2k}^2 + 1}{a_{2k+1}}$$

$$= 4a_{2k+1} + 4a_{2k} + a_{2k-1} = 2(2a_{2k+1} + a_{2k}) + 2a_{2k} + a_{2k-1}$$

$$= 2a_{2k+2} + a_{2k+1} \text{ as required.}$$

Similarly

$$a_{2k+4} = \frac{a_{2k+3}^2 - 1}{a_{2k+2}} = \frac{(2a_{2k+2} + a_{2k+1})^2 - 1}{a_{2k+2}} = 4a_{2k+2} + 4a_{2k+1} + \frac{a_{2k+1}^2 - 1}{a_{2k+2}}$$

$$= 4a_{2k+2} + 4a_{2k+1} + a_{2k} = 2(2a_{2k+2} + a_{2k+1}) + 2a_{2k+1} + a_{2k}$$

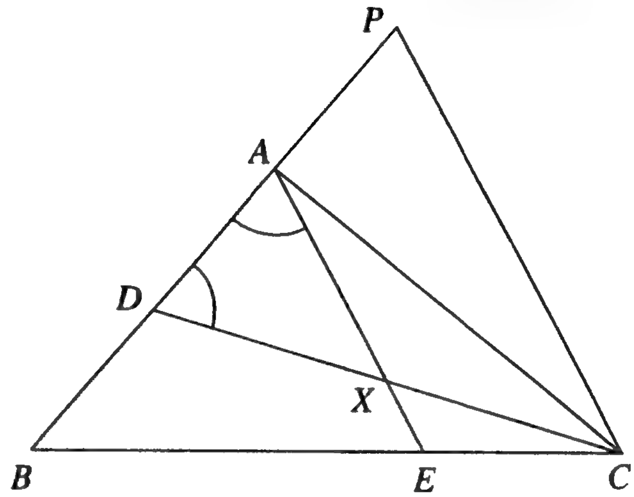
$$= 2a_{2k+3} + a_{2k+2} \text{ as required.}$$

5. In triangle ABC, D is the midpoint of AB and E is the point of trisection of BC nearer to C. Given that $\angle ADC = \angle BAE$ find $\angle BAC$.

Let X be the point of intersection of the lines AE and DC.

Solution 1 (John Drury, King's School Rochester & S. Lahiri, Colchester RGS)

Let P be the point on the produced line BA such that $|AP| = |AD| = |BD|$.

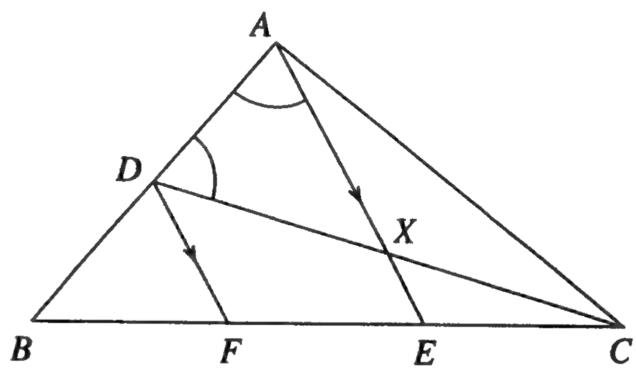


Since $|BP| : |BA| = |BC| : |BE| = 2 : 3$, PBC and ABE are similar triangles. It follows that $\angle BPC = \angle BAE = \angle CDA$.

Hence triangle CPD is isosceles. A is the midpoint of its base. So AC is perpendicular to DP. Hence $\angle BAC$ is 90°.

Solution 2

Draw a line from D parallel to AE. Let the point of intersection with BC be F.



Triangles BDF and BAE are similar. Since $|BD| = \frac{1}{2}|BA|$ it follows that $|BF| = \frac{1}{2}|BE|$ and so F is the other point of trisection of BC.

Triangles CEX and CFD are similar with $|CE| = \frac{1}{2}|CF|$. So $|CX| = \frac{1}{2}|CD| = |XD|$.

Triangle AXD is isosceles so $|AX| = |XD| = |XC|$. Consequently triangle AXC is also isosceles.

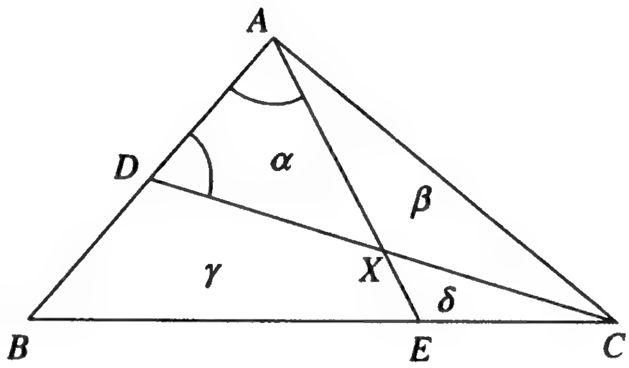
Let $\angle DAX = \theta°$. Then $\angle ADX = \theta°$ (given) so that $\angle AXD = 180° - 2\theta°$ (angle sum of triangle). Thus $\angle AXC = 2\theta°$ (angles on a straight line) and $\angle XAC = \frac{1}{2}(180 - 2\theta)° = (90 - \theta)°$ (triangle AXC is isosceles). So $\angle BAC = \angle DAX + \angle XAC = (\theta + 90 - \theta)° = 90°$. The angle BAC is 90°.

Alternative (Oliver Wicker, Cockermouth School & JJP Young, Nottingham High School)

Since D, A and C are all equidistant from X, there is a circle through D, A and C with centre X. DC is a diameter of the circle so $\angle BAC = \angle DAC$ is the angle subtended by a diameter at the circumference of a circle, 90°

Solution 3

Let α be the area of triangle AXD, β be the area of triangle AXC, γ be the area of $DXEB$ and δ be the area of triangle EXC.



Triangle ABE has twice the area of triangle AEC since it has twice the base $(|EC| = \frac{1}{2}|EB|)$ and the same perpendicular height. So

$$\alpha + \gamma = 2(\beta + \delta).$$

Similarly the triangle BXE has twice the area of triangle EXC and also triangle BDX has the same area as triangle ADX. Hence $\gamma = \alpha + 2\delta$.

Combining these equations yields $\alpha = \beta$. So the areas of AXD and AXC are equal. Using the common height from XD to A, it follows that the bases XD and XC have equal length.

The argument that $\angle BAC = 90°$ now follows as in Solution 2.

Solution 4

Let $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$. Then $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\mathbf{b}$ and $\overrightarrow{CE} = \frac{1}{3}\overrightarrow{CB} = \frac{1}{3}(\mathbf{b} - \mathbf{c})$. Hence $\overrightarrow{AE} = \overrightarrow{AC} + \overrightarrow{CE} = \frac{1}{3}\mathbf{b} + \frac{2}{3}\mathbf{c}$.

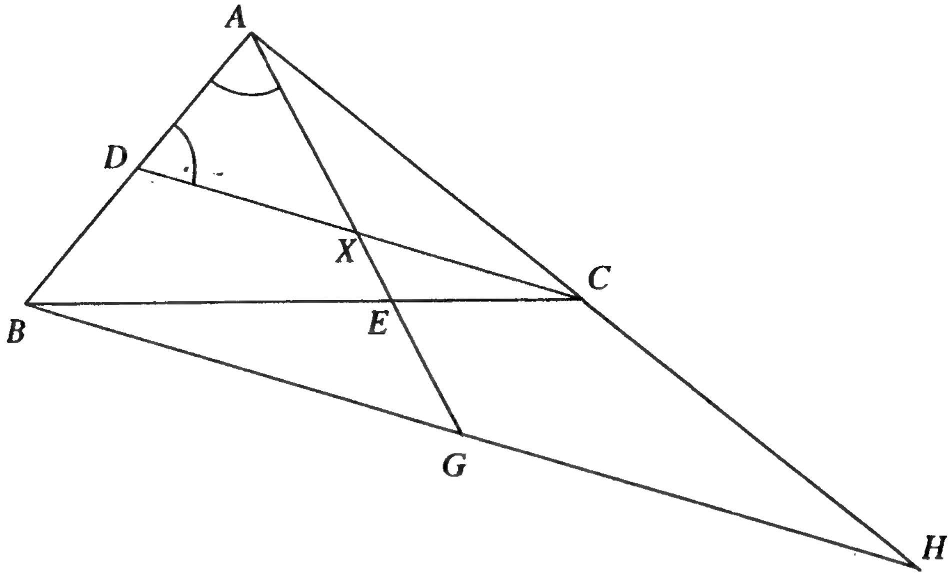
Taking A as the origin, the vector equation of line DC is $\mathbf{r} = \frac{1}{2}\mathbf{b} + \lambda(\mathbf{c} - \frac{1}{2}\mathbf{b})$ and the line AE is $\mathbf{r} = \mu(\frac{1}{3}\mathbf{b} + \frac{2}{3}\mathbf{c})$. These two lines meet at X when $\frac{1}{2}(1 - \lambda) = \frac{1}{3}\mu$ and $\lambda = \frac{2}{3}\mu$.

The solution to these simultaneous equations is $\lambda = \frac{1}{2}$ and $\mu = \frac{3}{4}$.

The value $\lambda = \frac{1}{2}$ implies that X is the midpoint of DC.

The argument that $\angle BAC = 90°$ now follows as in Solution 2.

Solution 5 (Seung-Tae Han, Hurstpierpoint College)



Let H be the point on the extended line AC such that $|AC| = |CH|$.

Then BC joins a vertex of the triangle ABH to the midpoint of the opposite side. So BC is a median of triangle ABH.

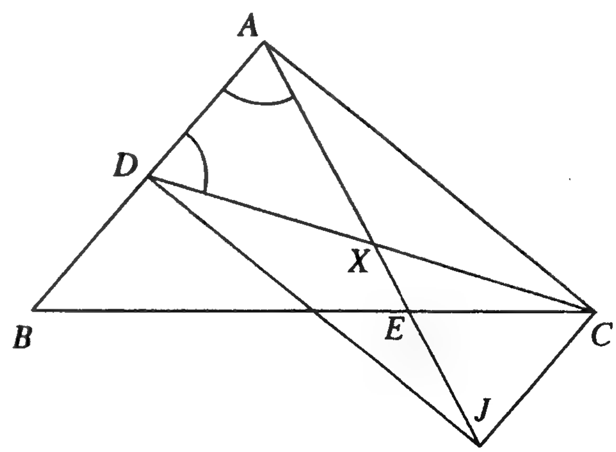
It follows that E is the centroid of the triangle ABH since it is two thirds of the way along the median BC from the vertex B. It follows that AE is also a median of the triangle and, extended, bisects BH at G, the point of intersection.

Since $|AD| : |AB| = |AC| : |AH| = 1 : 2$, DC is parallel to BH. So the extended line AE must cut DC and BH in the same ratio. Hence $|XD| = |XC|$.

The argument that $\angle BAC = 90°$ now follows as in Solution 2.

Solution 6 (R. Palmer, Haybridge High)

Construct a line through C parallel to AB. Let J be the point where this line meets the extension of the line AE.



Since AB and CJ are parallel, the triangles EAB and EJC are similar. It follows from the fact that E is a point of trisection of BC that the ratio of the lengths of corresponding sides of these triangles is 2 : 1.

Thus $|AD| = \frac{1}{2}|AB| = |JC|$. The opposite sides AD and CJ of $ACJD$ are parallel and of the same length so $ACJD$ is a parallelogram. X is the point where the diagonals of the parallelogram meet. Therefore X bisects the diagonals AJ and DC. Hence $|DX| = |XC|$.

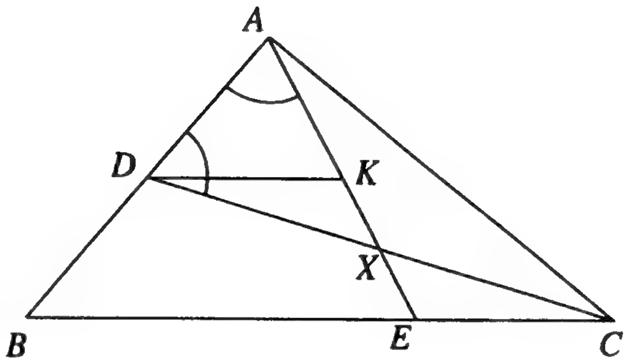
The argument that $\angle BAC = 90°$ now follows as in Solution 2.

Solution 7 (Liang Mu, Henrietta Barnett School).

Construct a line through D parallel to BC.

Let K be the point of intersection of this line with AE. Since DK is parallel to BE, triangles ADK and ABE are similar. Furthermore, since D bisects AB, the ratio of the lengths of corresponding sides is $1:2$. Therefore $|DK| = \frac{1}{2}|BE| = |EC|$.

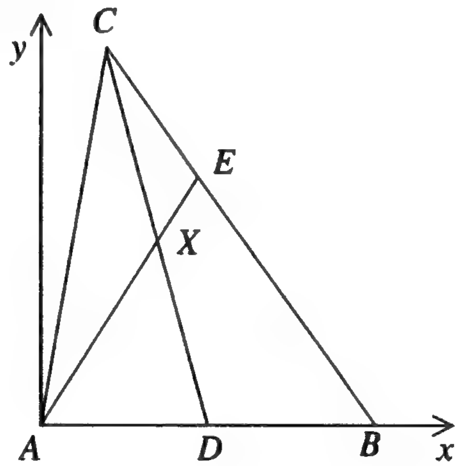
The angles $\angle XKD$ and $\angle XEC$ are equal, as are $\angle XCE$ and $\angle XDK$ (alternate angles). Since sides DK and EC are equal, triangles XKD and XEC are congruent (ASA) and $|XD| = |XC|$.



The argument that $\angle BAC = 90°$ now follows as in Solution 2.

Solution 8 (L. .J. Halliwell, Madras College)

Introduce coordinate axes with A as the origin. Let the x-axis be the line AB and the y-axis be the line perpendicular to AB, through A. In this coordinate system we may, without loss of generality, label the point D $(1, 0)$.



Since D is the midpoint of AB, B is the point $(2, 0)$. X is equidistant from A and D. Hence X lies on the line $x = \frac{1}{2}$.

Let the coordinates of C be (a, b). Since E is the point of trisection of BC closer to C, it has coordinates $\left(\frac{2a+2}{3}, \frac{2b}{3}\right)$. The line AE has equation $y = \frac{b}{a+1}x$ and intersects the line CD, which has equation $y = \frac{b(x-1)}{a-1}$, at $\left(\frac{a+1}{2}, \frac{b}{2}\right)$. But this is the point X which lies on the line $x = \frac{1}{2}$. Therefore $a = 0$, C lies on the y-axis and $\angle BAC = 90°$.

Solutions to the 1999 paper

1. I have four children. The age in years of each child is a positive integer between 2 and 16 inclusive and all four ages are distinct. A year ago the square of the age of the oldest child was equal to the sum of the squares of the ages of the other three. In one year's time the sum of the squares of the oldest and the youngest will be equal to the sum of the squares of the other two children.

   Decide whether this information is sufficient to determine their ages uniquely, and find all possibilities for their ages.

It turns out that there are two possibilities for the ages so the information does not determine the solution uniquely.

Algebraic Solution

We start by labelling the children's ages. Let their present ages be $a + 1$, $b + 1$, $c + 1$ and $d + 1$. Since their ages are distinct, and they are all aged between 2 and 16, we can assume that

$$1 \leqslant a < b < c < d \leqslant 15.$$

We note that $b \leqslant 13$ so that $(b - a) \leqslant 12$.

The information about their ages one year ago gives the equation

(i) $$d^2 = a^2 + b^2 + c^2,$$

and the information about their ages in one year's time yields the equation

(ii) $$(d + 2)^2 + (a + 2)^2 = (b + 2)^2 + (c + 2)^2.$$

Expanding brackets and subtracting the equations gives

(ii)–(i) $$4(a + d) + a^2 = 4(b + c) - a^2$$

which rearranges to

(iii) $$a^2 = 2(b + c - a - d).$$

We note that a must be even, since its square is even. Furthermore, since $d > c$,

$$a^2 = 2\big(b - a + (c - d)\big) < 2(b - a) < 24.$$

We conclude that there are only two possibilities for a, namely $a = 2$ and $a = 4$.

If $a = 4$ then, since $a^2 < 2(b - a)$, we have $2b > a^2 + 2a = 24$ so that $b > 12$. This forces $b = 13$, $c = 14$ and $d = 15$. This contradicts both the original equations, (i) and (ii). Hence there is no solution with $a = 4$.

If $a = 2$ then (iii) gives $b + c - d = 4$. Substituting $a = 2$ and $d = b + c - 4$ into (i) and simplifying gives

$$bc - 4b - 4c + 6 = 0$$

which is equivalent to

$$(b - 4)(c - 4) = 10.$$

Now $b - 4$ and $c - 4$ are integers between -2 and 10. There are only two possible ways to express 10 as a product of two integers in this range: $10 = 10 \times 1$ and $10 = 5 \times 2$. Since $c > b$ and $d = b + c - 4$, we have two possibilities

$$c = 14, b = 5, d = 15 \qquad \text{and} \qquad c = 9, b = 6, d = 11.$$

Both of these possibilities satisfy equations (i) and (ii).

Hence there are just two possibilities for the present ages of the children (3, 6, 15, 16) and (3, 7, 10, 12).

Exhaustive solution (Jennifer Austen, St Edward's School & Ravi Jain, Wilson's School)

There are several ways to eliminate possibilities at different stages of the proof above. The following proof includes no algebra except for the introduction of the initial equations,

$$d^2 = a^2 + b^2 + c^2, \tag{i}$$

$$(d + 2)^2 + (a + 2)^2 = (b + 2)^2 + (c + 2)^2. \tag{ii}$$

where $1 \leqslant a < b < c < d \leqslant 15$ are the children's ages one year ago.

We first concentrate on the equation (ii). We need to find two distinct pairs of squares of integers between 3 and 17 which have the same sum. Below is a table of sums $x^2 + y^2$ of pairs of distinct squares x^2 and y^2. We look for numbers which appear more than once in the table.

$y \backslash x$	4	5	6	7	8	9	10	11	12	13	14	15	16	17
3	25	34	45	58	73	90	109	**130**	153	178	**205**	234	**265**	298
4		41	52	65	80	97	116	137	160	**185**	212	241	272	**305**
5			61	74	89	106	125	146	169	194	**221**	**250**	281	314
6				85	100	117	136	157	180	**205**	232	261	292	**325**
7					113	**130**	149	170	193	218	245	274	**305**	338
8						145	164	**185**	208	233	260	289	320	353
9							181	202	225	**250**	277	306	337	370
10								**221**	244	269	296	**325**	356	389
11									**265**	290	317	346	377	410
12										313	340	369	400	433
13											365	394	425	458
14												421	452	485
15													481	514
16														545

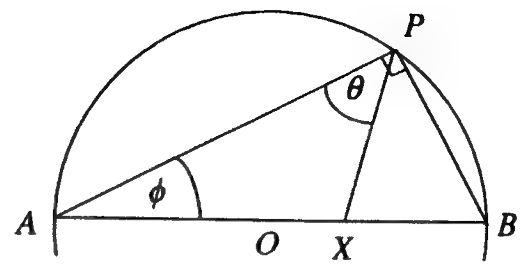
We have highlighted in the table eight numbers (130, 185, 205, 221, 250, 265, 305 and 325) which can be written as the sum of two different squares in two ways. These give all the sets of values which satisfy equation (ii), which are tabulated below.

	a	b	c	d	Does $d^2=a^2+b^2+c^2$?
130	1	5	7	9	No
185	2	6	9	11	Yes
205	1	4	11	12	No
221	3	8	9	12	No
250	3	7	11	13	No
265	1	9	10	14	No
305	2	5	14	15	Yes
325	4	8	13	15	No

We have also checked if these possibilities satisfy equation (i). In this way we have shown that there are only two possibilities for the ages of the children this year, (3, 7, 10, 12) and (3, 6, 15, 16).

2. A circle has diameter AB and X is a fixed point of AB lying between A and B. A point P, distinct from A and B, lies on the circumference of the circle. Prove that, for all possible positions of P, $\dfrac{\tan \angle APX}{\tan \angle PAX}$ remains constant.

We first note that $\angle APB = 90°$ (angle in a semicircle).
We label $\angle APX = \theta$ and $\angle PAX = \phi$.
Note that $\tan\phi = \dfrac{PB}{PA}$.

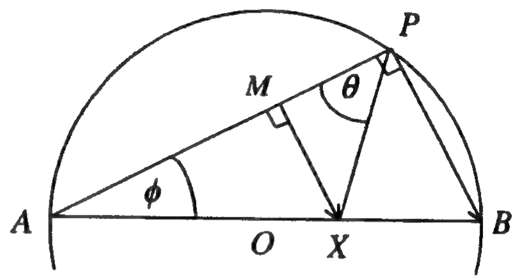


Geometric Solutions

For a geometric solution we need to construct a right-angled triangle so that we may express $\tan\theta$ in terms of the ratios of two lengths.

Solution (i)

Construct a line through X perpendicular to PB. Let M be its point of intersection with the line AP. Note that $\angle AMX = 90°$.



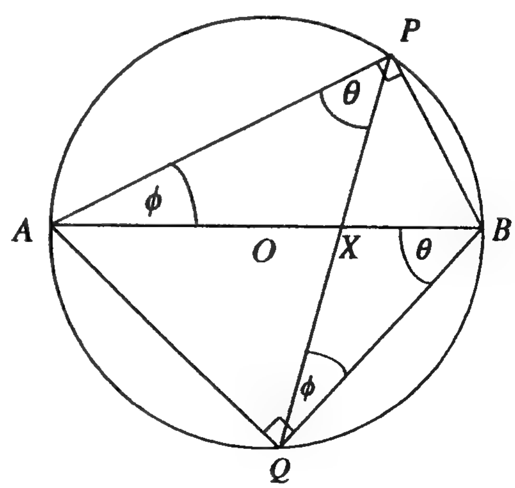
Then $\tan\theta = \dfrac{MX}{PM}$ and $\tan\phi = \dfrac{MX}{AM}$ so that $\dfrac{\tan\theta}{\tan\phi} = \dfrac{AM}{PM}$.

But triangles AMX and APB are similar so that $\dfrac{\tan\theta}{\tan\phi} = \dfrac{AM}{PM} = \dfrac{AX}{BX}$. Since X is a fixed point, this ratio is constant as required.

Solution (ii)

Extend the line PX to meet the circumference again at Q.

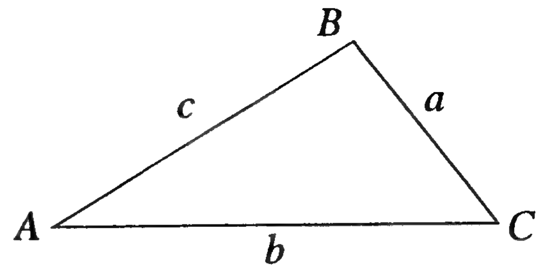
Since AB is a diameter $\angle AQB = 90°$. Also $\angle APQ$ and $\angle ABQ$ are subtended by the same chord, AQ, so that $\angle ABQ = \theta$. Similarly $\angle BQP = \angle BAP = \phi$. It is easily checked that triangles APX and QBX are similar, as are triangles AXQ and BXP.



Now $\tan\theta = \dfrac{AQ}{BQ}$ and $\tan\phi = \dfrac{PB}{PA}$ so that $\dfrac{\tan\theta}{\tan\phi} = \dfrac{PA}{BQ} \times \dfrac{AQ}{PB} = \dfrac{AX}{XQ} \times \dfrac{XQ}{BX} = \dfrac{AX}{BX}$ by similar triangles. This ratio is constant since X is a fixed point.

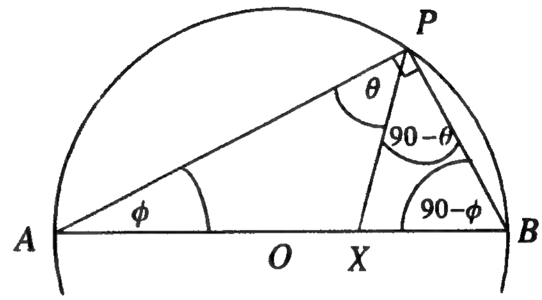
Trigonometric Solution

In this solution we use the Sine Rule which states that in any triangle labelled with lengths a, b and c and angles A, B and C as in the picture,



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This rearranges to $\dfrac{\sin A}{\sin B} = \dfrac{a}{b}$.



Note first that $\angle XPB = 90 - \theta$ and $\angle XBP = 90 - \phi$. Hence,

$$\begin{aligned}\frac{\tan\theta}{\tan\phi} &= \frac{\sin\theta}{\cos\theta} \div \frac{\sin\phi}{\cos\phi} = \frac{\sin\theta}{\cos\theta} \times \frac{\cos\phi}{\sin\phi} \\ &= \frac{\sin\theta}{\sin\phi} \times \frac{\sin(90-\phi)}{\sin(90-\theta)} \\ &= \frac{AX}{PX} \times \frac{PX}{BX} \text{ (Sine Rule)} \\ &= \frac{AX}{BX}.\end{aligned}$$

As X is a fixed point, this ratio is constant, as required.

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3. Determine a positive constant c such that the equation

$$xy^2 - y^2 - x + y = c$$

has precisely three solutions (x, y) in positive integers.

Tabular Solution (Stephen Brooks, Abingdon School)

We label the function $f(x, y) = xy^2 - y^2 - x + y$.

We note that for positive integers x and y

$$f(x+1, y) - f(x, y) = y^2 - 1 > 0, \text{ when } y > 1$$

$$f(x, y+1) - f(x, y) = 2xy + x - 2y = (x-1)(2y+1) + 1 > 0.$$

The following is a table of values of $xy^2 - y^2 - x + y$ for different integers x and y. The inequalities above establish that, except for the first row where $y = 1$, these values increase along the rows and down the columns.

$y \backslash x$	1	2	3	4	5	6	7	8	9	10
1	0	0	0	0	0	0	0	0	0	0
2	1	4	7	**10**	13	16	19	22	25	28
3	2	**10**	18	26	34	42	50	58	66	74
4	3	18	33	48	63	78	93	108	123	138
5	4	28	52	76	100	124	148	172	196	220
6	5	40	75	110	145	180	215	250	285	320
7	6	54	102	150	198	246	294	342	390	438
8	7	70	133	196	259	322	385	448	511	574
9	8	88	168	248	328	408	488	568	648	728
10	9	108	207	306	405	504	603	702	801	900
11	**10**	130	250	370	490	610	730	850	970	1090
12	11	154	297	440	583	726	869	1012	1155	1298
13	12	180	348	516	684	852	1020	1188	1356	1524

By inspection we can see that the number 10 can only occur in the three places highlighted. Hence $c = 10$ is a solution.

Note

$c = 10$ is not the only solution. There are many more values for c which satisfy the criteria of the question. Below is a table with values of suitable $c \leqslant 100$ with their corresponding triplets of solutions (x, y) to $xy^2 - y^2 - x + y = c$.

c	10	18	28	34	40	52	58	70	76	82	88	100
	(1, 11)	(1, 19)	(1, 29)	(1, 35)	(1, 41)	(1, 53)	(1, 59)	(1, 71)	(1, 77)	(1, 83)	(1, 89)	(1, 101)
	(2, 3)	(2, 4)	(2, 5)	(5, 3)	(2, 6)	(3, 5)	(8, 3)	(2, 8)	(4, 5)	(11, 3)	(2, 9)	(5, 5)
	(4, 2)	(3, 3)	(10, 2)	(12, 2)	(14, 2)	(18, 2)	(20, 2)	(24, 2)	(26, 2)	(28, 2)	(30, 2)	(34, 2)

Algebraic Solution

We start with some observations.

- If $x = 1$ then the equation yields $y = c + 1$. Hence $(x, y) = (1, c + 1)$ is one solution, whatever the value of c.
- The equation can be factorised to give $(y - 1)\big(y(x - 1) + x\big) = c$. The second bracket is positive for all positive integers x and y. Therefore $y > 1$ and $y - 1$ is a divisor of c.

Let $c = ab$ be some factorisation of c into two positive integers. If there is a solution with $y - 1 = a$ then, solving $(y-1)\big(y(x-1)+x\big) = c$ for x, we have

$$x = \frac{b + a + 1}{a + 2} = 1 + \frac{b - 1}{a + 2}.$$

The trivial factorisations of c give the two solutions $(x, y) = \left(1 + \frac{1}{3}(c - 1),\ 2\right)$, $(a = 1,\ b = c)$ and $(x, y) = (1, c + 1)$, $(a = c, b = 1)$ provided $c \equiv 1 \bmod 3$.

Since x is an integer then $(a + 2)$ must be a divisor of $(b - 1)$, or $b \equiv 1 \bmod (a + 2)$. If $b \neq 1$ then $a + 2 \leqslant b - 1$ so that $a \leqslant b + 3$. This suggests a search for values for c which are the product of two primes, $p_1 < p_2$, three or more apart, satisfying $p_2 \equiv 1 \bmod (p_1 + 2)$ and $c = p_1 p_2 \equiv 1 \bmod 3$. In this way only three of the four possible factorisations of c into pairs, $(a = 1,$ $b = c)$, $(a = c,\ b = 1)$, $(a = p_1,\ b = p_2)$ could yield solutions, since $(a = p_2, b = p_1)$ does not satisfy $a < b$.

The prime pair $p_1 = 2,\ p_2 = 5$ satisfies these criteria. The equation $xy^2 - y^2 - x + y = 10$ has exactly three solutions in the integers, $(x, y) = (1,\ 11)$, $(x, y) = (4,\ 2)$, $(x, y) = (2,\ 3)$. Hence $c = 10$ is a solution.

---

4. Any positive integer m can be written uniquely in base 3 form as a string of 0's, 1's and 2's (not beginning with a zero). For example

$$98 = (1 \times 81) + (0 \times 27) + (1 \times 9) + (2 \times 3) + (2 \times 1) = (10122)_3.$$

Let $c\,(m)$ denote the sum of the cubes of the digits of the base 3 form of m; thus, for instance

$$c\,(98) = 1^3 + 0^3 + 1^3 + 2^3 + 2^3 = 18.$$

Let n be any fixed positive integer. Define the sequence (u_r) by

$$u_1 = n \qquad \text{and} \qquad u_r = c\left(u_{r-1}\right) \qquad \text{for } r \geqslant 2.$$

Show that there is a positive integer r for which $u_r = 1,\ 2$ or 17.

The digits of a base 3 number are at most 2 and so their cubes are at most 8. If a number, q, has k digits in base 3 then the sum of the cubes of its base 3 digits satisfies $c\,(q) \leqslant 8k$. Furthermore $q \geqslant 3^{k-1}$ since its leading digit is not zero.

We can check by substitution that $3^{k-1} = 81 \times 3^{k-5} > 8k$ when $k = 5$. The same inequality follows for integers $k > 5$ since the left hand side, $81 \times 3^{k-5}$, increases by at least 81 each time k is increased by 1, whereas $8k$ increases by just 8. Hence it is established that $q \geqslant 3^{k-1} > 8k \geqslant c\,(q)$ when $q \geqslant 81$.

If $27 \leqslant q \leqslant 53$ then $c\,(q) \leqslant 1^3 + 2^3 + 2^3 + 2^3 = 25 < q$, since q has four digits, with first digit equal to 1 in base 3.
If $54 \leqslant q \leqslant 80$ then $c\,(q) \leqslant 2^3 + 2^3 + 2^3 + 2^3 = 32 < q$ since q has four digits in base 3.

We have established above that, when $u_r \geqslant 27$, $u_{r+1} = c\,(u_r) < u_r$, i.e. the sequence will decrease until it reaches a number less than 27. There must be a term, u_s, in the sequence with $u_s < 27$. We need only check to see what

happens to the terms in the sequence after u_s. Below is a table of possibilities.

u_s	In Base 3	u_{s+1}	u_{s+2}	u_{s+3}	u_{s+4}	u_s	In Base 3	u_{s+1}	u_{s+2}	u_{s+3}	u_{s+4}
26	222	24	16	10	2	13	111	3	1		
25	221	17				12	110	2			
24	220	16	10	2		11	102	9	1		
23	212	17				10	101	2			
22	211	10	2			9	100	1			
21	210	9	1			8	22	16	10	2	
20	202	16	10	2		7	21	9	1		
19	201	9	1			6	20	8	16	10	2
18	200	8	16	10	2	5	12	9	1		
17	122					4	11	2			
16	121	10	2			3	10	1			
15	120	9	1			2	2				
14	112	10	2			1	1				

Since 1, 2 or 17 appears in each of the sequences we have proved the required result.

5. Consider all functions f from the positive integers to the positive integers such that

   (i) for each positive integer m, there is a unique positive integer n such that $f(n) = m$;

   (ii) for each positive integer n, we have

$$f(n+1) \text{ is either } 4f(n) - 1 \text{ or } f(n) - 1.$$

   Find the set of positive integers p such that $f(1999) = p$ for some function f with properties (i) and (ii).

We consider the sequence $f(1), f(2), f(3), \ldots$. Condition (i) states that every positive integer appears exactly once in this sequence. Condition (ii) states that terms in the sequence are either bigger, $f(n+1) = 4f(n) - 1$, or exactly one smaller, $f(n+1) = f(n) - 1$, than their predecessor. If a term, $f(n+1)$, is bigger than its predecessor, $f(n)$, then every term which follows in the sequence is also bigger than $f(n)$. This is because successive terms in the sequence can only decrease in steps of 1. If the sequence were to decrease to $f(n)$ or below then there would be two terms in the sequence equal to $f(n)$. Since every positive integer must appear in the sequence exactly once, it follows that $f(n+1)$ must equal $f(n) - 1$ if $f(n) - 1$ does not appear in the sequence

before $f(n)$. Otherwise $f(n+1) = 4f(n) - 1$.

The number 1 must appear in the sequence. If it is not the first term then it must be preceded by 2. Also 1 must be followed in the sequence by 3 since there are no smaller positive integers. Hence, if 2 precedes 1, then 2 must be the first term.

We have established that either $f(1) = 1$ or $f(1) = 2$. We have also established that, once the first term in the sequence is decided then the rest of the sequence is fully determined.

If $f(1) = 1$ then $f(2) = 3, f(3) = 2, f(4) = 7, f(5) = 6, \ldots$

This sequence is described exactly by the algebraic rule

$$f(2^n + k) = 2^{n+1} - (k+1), \text{ where } n = 0, 1, 2, \ldots, \text{and } 0 \leqslant k \leqslant 2^n - 1.$$

We can prove this formula for $f(t)$ by strong induction.

It gives the correct result for $t = 1 = 2^0 + 0$ since $f(1) = 1$ and $2^{n+1} - (k+1) = 2^1 - 1 = 1$.

Now we assume that the formula is true for all $t \leqslant s$, and consider when $t = s + 1$. If $s = 2^n + k$, with $0 \leqslant k \leqslant 2^n - 2$, then $f(s) = f(2^n + k) = 2^{n+1} - (k+1)$ and $f(s+1) = f(s) - 1$, since, by the inductive hypothesis, the values between $2n$ and $2^{n+1} - (k+2)$ have not appeared in the sequence $f(1), f(2), \ldots, f(s)$. Thus $f(s+1) = f(2^n + k + 1) = 2^{n+1} - (k+1) - 1 = 2^{n+1} - ((k+1)+1)$. If $s = 2^n + k$, with $k = 2^n - 1$, then $f(s) = 2^n$ and $f(s+1) = 4f(s) - 1$, since, by the inductive hypothesis, the value $2^n - 1 = f(2^{n-1})$ already appears in the sequence. Hence we have $f(s+1) = f(2^{n+1}) = 4 \times 2^n - 1 = 2^{n+2} - 1$. In either case the formula gives the correct value for the term in the sequence. The formula is correct for $t = 1$ and, if we assume that it is correct for all values of $t \leqslant s$ then it is true for $t = s + 1$. Hence we have proved that the formula is correct for all positive integers t by induction.

Now $1999 = 2^{10} + 975$. Hence $f(1999) = 1072$.

If $f(1) = 2$ then $f(2) = 1, f(3) = 3, f(4) = 11, f(5) = 10, \ldots f(11) = 4, f(12) = 15, \ldots f(15) = 12, f(16) = 47, \ldots$

This sequence is described exactly by the algebraic rule

$f(4^n + k) = 3 \times 4^n - (k+1)$, where $n = 0, 1, 2, \ldots$, and $0 \leqslant k \leqslant 2 \times 4^n - 1$

$f(3 \times 4^n + k) = 4^{n+1} - (k+1)$, where $n = 0, 1, 2, \ldots$, and $0 \leqslant k \leqslant 4^n - 1$.

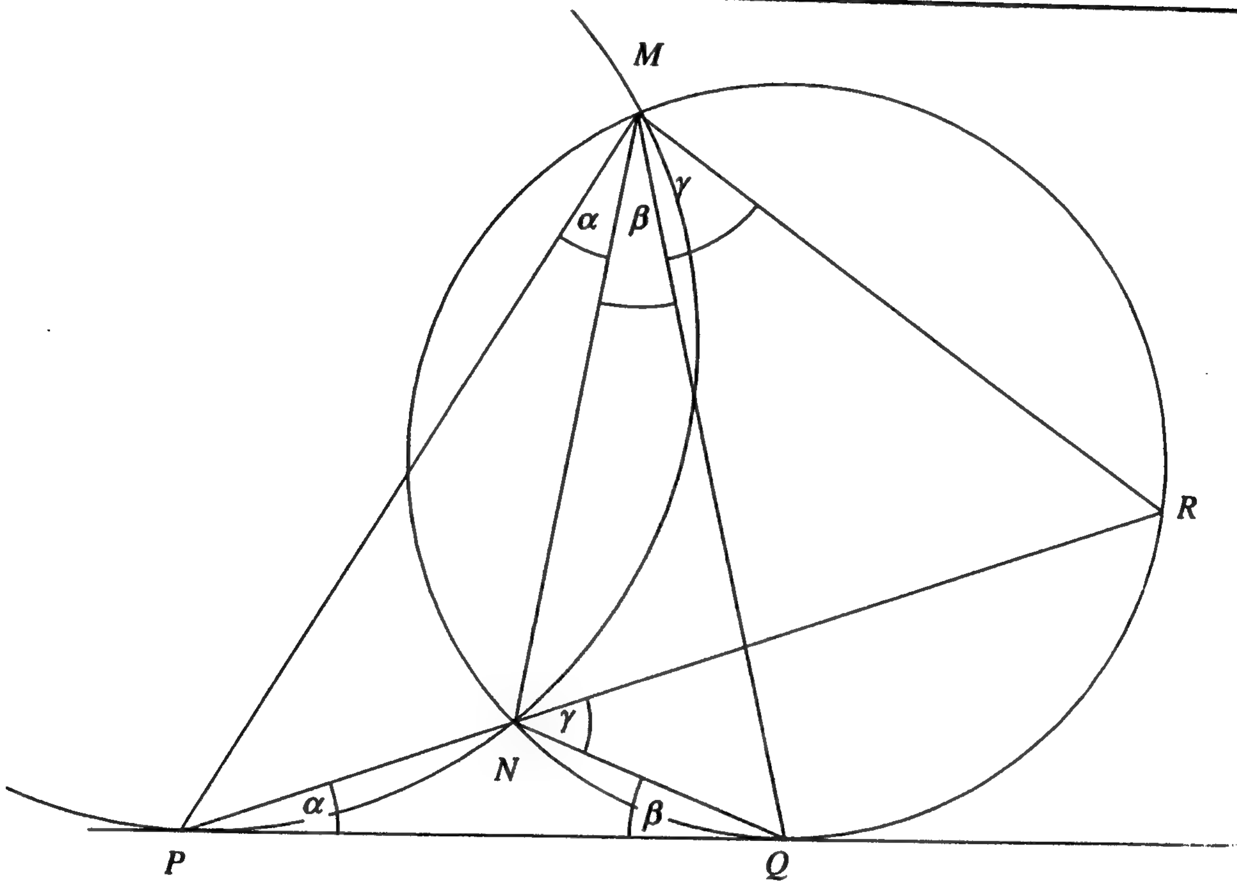
This formula can also be proved by induction.

Now $1999 = 4^5 + 975$. Hence $f(1999) = 3 \times 1024 - 976 = 2096$.

The set of positive integers p such that $f(1999) = p$ is $\{1072, 2096\}$.

Solutions to the 2000 paper

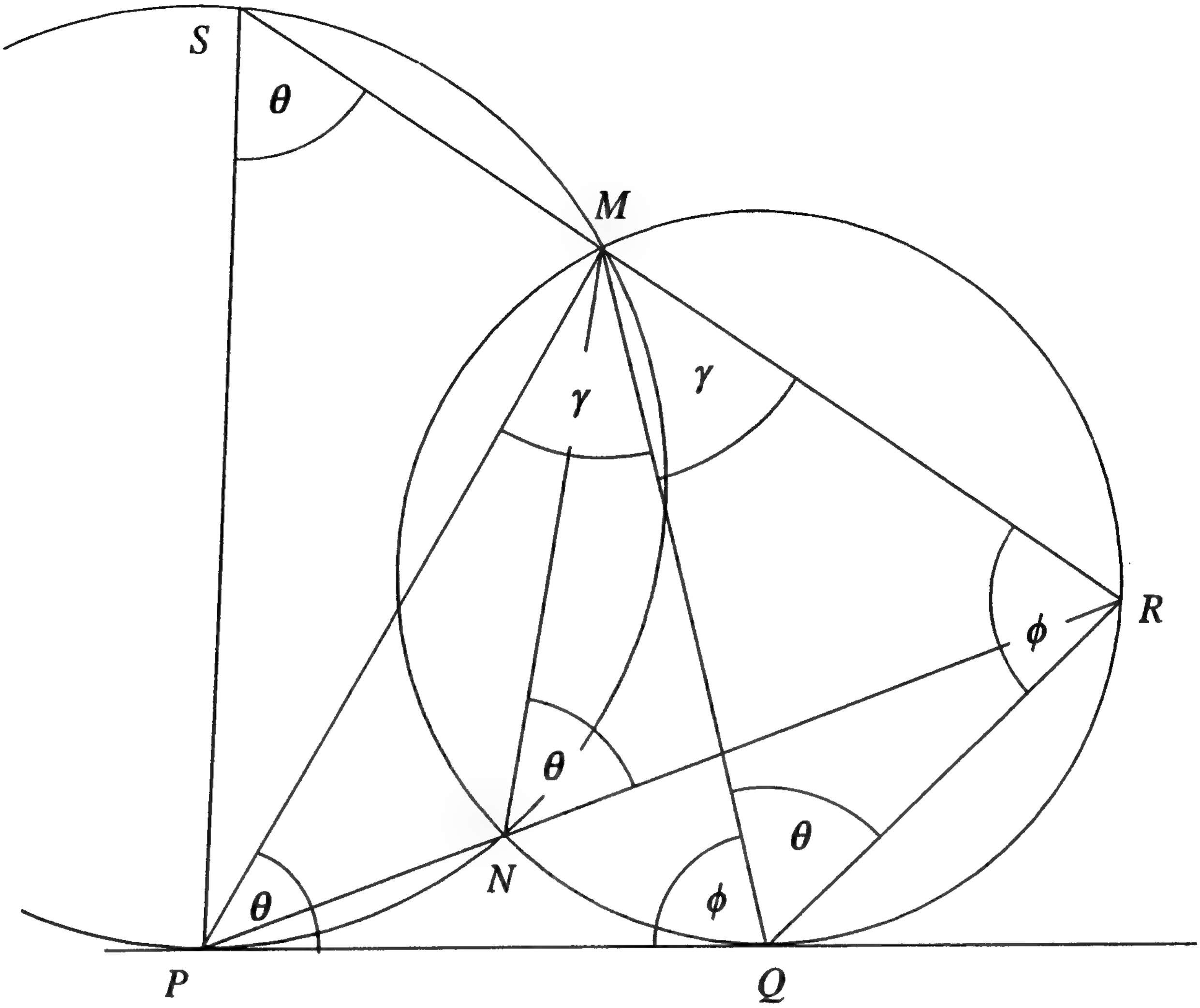
1. Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects angle PMR.



Join the points M and N.

Let $\angle PMN = \alpha$, $\angle NMQ = \beta$ and $\angle QMR = \gamma$.
Then $\angle QPN = \alpha$ and $\angle PQN = \beta$ by the alternate segment theorem applied respectively to the circles C_1 and C_2.
Furthermore $\angle QNR = \gamma$ since angles QMR and QNR are subtended by the same chord, QR, in the circle C_2.
The external angle, $\angle QNR$, of the triangle $\angle PNQ$, is equal to the sum of the internal opposite angles, $\angle QPN$ and $\angle PQN$.
Hence $\alpha + \beta = \gamma$ and $\angle PMQ = \angle QMR$ as required.

Alternative Solution (P. S. Cooper, SEEVIC)



Produce the chord RM to meet C_1 at S. Join the lines MN, SP and RQ.

Let $\angle MPQ = \theta$, $\angle MQP = \phi$ and $\angle PMQ = \gamma$. Note that $\theta + \phi + \gamma = 180°$ (angle sum of triangle PQM).

By the alternate segment theorem $\angle PSM = \theta$, and $\angle QRM = \phi$.

Now $\angle RNM = \angle PSM = \theta$ (exterior angle of cyclic quadrilateral $SMNP$) and $\angle MQR = \angle MNR = \theta$ (angles subtended by the same chord, MR).

Hence $\angle QMR = 180° - \theta - \phi = \gamma$ (angle sum of triangle MQR).

We have now shown that $\angle QMR = \angle PMQ$ and the proof that MQ bisects $\angle PMR$ is complete.

2. Show that, for every positive integer n,

$$121^n - 25^n + 1900^n - (-4)^n$$

is divisible by 2000.

We first note that, since $2000 = 2^4 \times 5^3$ and the highest common factor of 2^4 and 5^3 is 1, it suffices to prove that the given expression is divisible by $16 = 2^4$ and $125 = 5^3$ separately.

Method 1 (using factorisation)

We also note that any expression of the form $(x^n - y^n)$, where n is a positive integer, has $(x - y)$ as a factor:

$$(x^n - y^n) = (x-y)\left(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + x^{n-r-1}y^r + \ldots + y^{n-1}\right).$$

By pairing off terms in the given expression, we can deduce quickly that it is divisible by 16 and 125:

$$\begin{aligned} 121^n - 25^n + 1900^n - (-4)^n &= \left(121^n - 25^n\right) + \left(1900^n - (-4)^n\right) \\ &= 96c + 1904d, \text{ where } c \text{ and } d \text{ are integers} \\ &= 16(6c + 119d) \end{aligned}$$

$$\begin{aligned} 121^n - 25^n + 1900^n - (-4)^n &= \left(121^n - (-4)^n\right) + \left(1900^n - 25^n\right) \\ &= 125a + 1875b, \text{ where } a \text{ and } b \text{ are integers} \\ &= 125(a + 15b). \end{aligned}$$

Method 2 (using modular arithmetic)

$$121 \equiv 25 \pmod{16} \Rightarrow 121^n \equiv 25^n \pmod{16}$$

$$1900 \equiv -4 \pmod{16} \Rightarrow 1900^n \equiv (-4)^n \pmod{16}.$$

$$\text{Hence } 121^n - 25^n + 1900^n - (-4)^n \equiv 0 \pmod{16}.$$

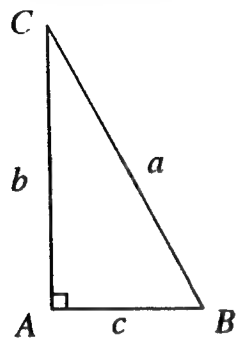
$$\text{Similarly } 121 \equiv -4 \pmod{125} \Rightarrow 121^n \equiv (-4)^n \pmod{125}$$

$$1900 \equiv 25 \pmod{125} \Rightarrow 1900^n \equiv 25^n \pmod{125}.$$

$$\text{Hence } 121^n - 25^n + 1900^n - (-4)^n \equiv 0 \pmod{125}.$$

In each case we have shown that $121^n - 25^n + 1900^n - (-4)^n$ is divisible by 16 and 125 and hence that it is divisible by 2000.

3. Triangle ABC has a right-angle at A. Among all points P on the perimeter of the triangle, find the position of P such that $AP + BP + CP$ is minimised.

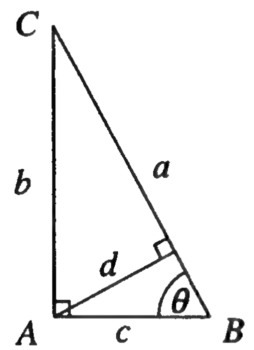


We start by labelling the lengths BC, AC, AB by a, b, c respectively as shown on the diagram.

We consider separately the possibilities: P is on AC, P is on AB, P is on BC.

If P is on AC, then $AP + CP = b$ so minimising $AP + BP + CP$ amounts to minimising BP. This occurs when P is at A, since the perpendicular distance of a point from a line is the least. The minimum of $AP + BP + CP$ in this case is $b + c$.

Similarly if P is on AB, then $AP + BP = c$ so that minimising $AP + BP + CP$ amounts to minimising CP. For the same reason, this also occurs when P is at A and the minimum of $AP + BP + CP$ is once again $b + c$.



Now suppose that P is on BC. Then $BP + CP = a$ so that minimising $AP + BP + CP$ amounts to minimising AP. This happens when AP is perpendicular to BC. This length is labelled d in the diagram on the left. In this case the minimum of $AP + BP + CP$ is $a + d$.

The question now boils down to a comparison of $a + d$ with $b + c$.

Method 1

We note first that we can find the area of the triangle ABC in two ways, yielding $\frac{1}{2}bc = \frac{1}{2}ad$, which gives us $2bc = 2ad$.

By Pythagoras' Theorem $a^2 = b^2 + c^2$, and, since $d^2 > 0$,

we have $a^2 + d^2 > b^2 + c^2$.

Hence $\quad a^2 + 2ad + d^2 > b^2 + 2bc + c^2$

i.e. $\quad (a + d)^2 > (b + c)^2$

i.e. $\quad a + d > b + c$ since all quantities are non-negative.

Method 2

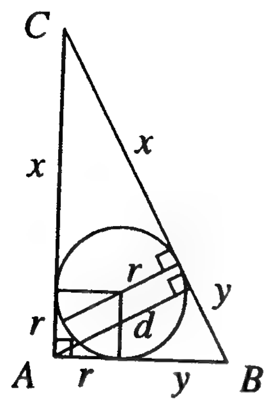
Let angle $ABC = \theta$. Then $d = c \sin\theta$ and $b = a \sin\theta$. Since BC is the hypotenuse, $a > c$ and, since $0° < \theta < 90°$, $1 - \sin\theta > 0$. Hence

$$a(1 - \sin\theta) > c(1 - \sin\theta)$$

i.e. $$a + c\sin\theta > c + a\sin\theta$$

i.e. $$a + d > b + c.$$

Method 3 (S. E. Thomas, Kingston Grammar School)



Consider the inscribed circle of triangle ABC.

Note that the distances from a point outside a circle along the two tangents to where they meet the circle are equal. We label the distances from points B and C along the sides of the triangle to the circle x and y respectively. We also label the radius of the circle r. This is the distance from the point A to the circle.

Then $b + c = x + y + 2r$ and $a + d = x + y + d$.
It is clear, for example by considering similar triangles, that $d > 2r$ and hence that $a + d > b + c$.

Conclusion

We conclude that the minimum of $AP + BP + CP$ is $b + c$ and occurs when P is at A.

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4. For each positive integer k, define the sequence $\{a_n\}$ by
$$a_0 = 1, \qquad a_n = kn + (-1)^n a_{n-1} \text{ for each } n \geqslant 1.$$
Determine all values of k for which 2000 is a term of the sequence.

We prove that the terms in the sequence take the form
$$\begin{aligned} a_{4m} &= 4mk + 1 \\ a_{4m+1} &= k - 1 \\ a_{4m+2} &= (4m + 3)k - 1 \\ a_{4m+3} &= 1 \end{aligned} \qquad (m > 0).$$
We start by considering the odd terms in the sequence.
If n is odd then, from the definition of the sequence,
$$\begin{aligned} a_{n+2} &= k(n + 2) + (-1)^{n+2} a_{n+1} \\ &= k(n + 2) - a_{n+1} \\ &= k(n + 2) - \left(k(n + 1) + (-1)^{n+1} a_n\right) \\ &= k(n + 2) - k(n + 1) - a_n \\ &= k - a_n \end{aligned}$$
Applying this formula twice, we have $a_{n+4} = k - a_{n+2} = k - (k - a_n) = a_n$.
Since $a_1 = k - 1$, the odd terms in the sequence are given by $a_{4m+1} = k - 1, a_{4m+3} = 1$.
Having established this relationship, the formulae for a_{4m+2} $(m \geqslant 0)$ and a_{4m} $(m \geqslant 0)$ follow from the definition of the sequence:

$$a_{4m+2} = k(4m+2) + (-1)^{4m+2}(k-1) \qquad a_{4m} = k(4m) + (-1)^{4m} a_{4(m-1)+3}$$
$$= k(4m+2) + (k-1) \qquad = 4mk + 1$$
$$= (4m+3)k - 1$$

We conclude that 2000 appears in the sequence if

- $2000 = 4mk + 1$ or $k = \dfrac{1999}{4m}$. This does not yield any integer solutions for k since 4 is not a factor of 1999.
- $2000 = k - 1$. This yields just one solution, $k = 2001$.
- $2000 = (4m + 3)k - 1$ or $k = \dfrac{2001}{4m + 3}$. The prime factors of 2001 are $2001 = 3 \times 23 \times 29$. The factors of 2001 are 1, 3, 23, 29, $3 \times 23 = 69$, $3 \times 29 = 87$, $23 \times 29 = 667$ and 2001. Of these only 3, 23, 87 and 667 have the form $4m + 3$, where m is an integer. These give 667, 87, 23, 3 as the only possible values of k in this case.
- $2000 = 1$. This gives no solutions.

The values of k for which 2000 is a term of the sequence are 3, 23, 87, 667, 2001.

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5. The seven dwarfs decide to form four teams to compete in the Millennium Quiz. Of course, the sizes of the teams will not all be equal. For instance, one team might consist of Doc alone, one of Dopey alone, one of Sleepy, Happy and Grumpy as a trio, and one of Bashful and Sneezy as a pair. In how many ways can the four teams be made up?

   (The order of the teams or of the dwarfs within the teams does not matter, but each dwarf must be in exactly one of the teams.)

   Suppose Snow White agreed to take part as well. In how many ways could the teams then be formed?

We assume throughout that a team has at least one member and that all the players are assigned to a team.

Method 1 – Recurrence Solution

Let $n_{d,t}$ be the number of ways of forming t teams from d players, with $t \leqslant d$. In this notation, the question asks us to find $n_{7,4}$ and $n_{8,4}$.

There is only ever one way to form 1 team from d players, comprising all the players, so that $n_{d,1} = 1$.

There is only ever one way to form d teams from d players, where each player constitutes a complete team, so that $n_{d,d} = 1$.

Let us now single out a player, Bashful say, when we want to form t teams with d players (including Bashful), $1 < t < d$. There are 2 cases.

Bashful could be a team on his own. The number of ways of arranging this is $n_{d-1,t-1}$, the number of ways to form the remaining $t - 1$ teams from the other $d - 1$ players.

Alternatively Bashful could be in a larger team. We start by considering the other players. They can be placed into t teams, in $n_{d-1,t}$ ways. Now Bashful is added to one of these t teams. Each arrangement of the other $d - 1$ players into t teams and each addition of Bashful to one of these teams gives rise to a different arrangement of t teams with d players, yielding a total of $t \times n_{d-1,t}$ ways in which Bashful can be in a team with other players.

	$d =$	1	2	3	4	5	6	7	8
$t = 1$		1	1	1	1	1	1	1	1
$t = 2$			1	3	7	15	31	63	127
$t = 3$				1	6	25	90	301	966
$t = 4$					1	10	65	350	1701

Hence we have the recurrence relationship $n_{d,t} = n_{d-1,t-1} + t \times n_{d-1,t}$ which is used to calculate the values in the table above.

We get the answers $n_{7,4} = 350$ and $n_{8,4} = 1701$.

Method 2 – Counting Solution

This solution relies first upon finding out the possible sizes of the teams and then counting up the number of ways of placing the dwarfs into teams of these sizes. It is essential that a method of counting is chosen so that each possible arrangement of teams is counted exactly once.

7 Dwarfs

There must always be at least one team of size 1 since $4 \times 2 = 8$ and there can be at most three teams of size 1. The largest possible team size is 4. Listing possible team sizes in descending order we get

4, 1, 1, 1 which can be done in ${}^7C_4 = 35$ ways. (Choose the team of 4 first, then the singles are made up from the remaining players.)

3, 2, 1, 1 which can be done in ${}^7C_3 \times {}^4C_2 = 210$ ways. (Choose the team of 3 first, then the team of 2 from the remaining 4, then the singles are made up from the remaining players.)

2, 2, 2, 1 which can be done in $\dfrac{{}^7C_2 \times {}^5C_2 \times {}^3C_2}{3!} = 105$ ways. (Choose the first team of 2 from 7, the second from 5, the third from 3, leaving the final team of 1. However the three teams of 2 could have been picked in any order which accounts for the division by 3! .)

This gives a total of 350 ways.

7 Dwarfs and Snow White

There can be at most three teams of size 1. The largest possible team size is 5. Listing possible team sizes in descending order we get

5, 1, 1, 1 which can be done in ${}^8C_5 = 56$ ways

4, 2, 1, 1 which can be done in ${}^8C_4 \times {}^4C_2 = 420$ ways

3, 3, 1, 1 which can be done in $\dfrac{{}^8C_3 \times {}^5C_3}{2!} = 280$ ways

3, 2, 2, 1 which can be done in ${}^8C_3 \times \dfrac{{}^5C_2 \times {}^3C_2}{2!} = 840$ ways

2, 2, 2, 2 which can be done in $\dfrac{{}^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2}{4!} = 105$ ways.

This gives a total of 1701 ways.

This booklet contains the BMO Round 1 question papers for the years 1997 to 2000. It also contains the corresponding answers and solutions.

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